STEP II 2002 Q1

$$\int_{\pi/6}^{\pi/4} \frac{1}{1-\cos 2\theta}\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \frac{1}{1-(1-2\sin^2\theta)}\, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/4} \operatorname{cosec}^2\theta \, d\theta$$

$$= \frac{1}{2}\Big[-\cot x\Big]_{\pi/6}^{\pi/4}$$

$$= \frac{1}{2}\left(\sqrt{3}-1\right)$$

$$\int_{\sqrt{3}/2}^{1} \frac{1}{1-\sqrt{1-x^2}}\, dx \qquad x = \sin 2\theta$$

$$dx = 2\cos 2\theta \, d\theta$$

$$\int_{\pi/6}^{\pi/4} \frac{1}{1-\cos 2\theta}\cdot 2\cos 2\theta \, d\theta$$

$$= -2\int_{\pi/6}^{\pi/4} 1 - \frac{1}{1-\cos 2\theta}\, d\theta$$

$$= -2\left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2}\right) \qquad \text{(by above)}$$

$$= \sqrt{3} - 1 - \pi/6$$

$$\int_{1}^{2/\sqrt{3}} \frac{1}{y(y-\sqrt{y^2-1^2})}\, dy \qquad y = \frac{1}{x},\ dy = \frac{-1}{x^2}\, dx$$

$$= \int_{1}^{\sqrt{3}/2} \frac{1}{\frac{1}{x}\left(\frac{1}{x}-\sqrt{\frac{1}{x^2}-1}\right)} \cdot \frac{-1}{x^2}\, dx$$

$$= \int_{\sqrt{3}/2}^{1} \frac{1}{1-\sqrt{1-x^2}}\, dx = \sqrt{3} - 1 - \frac{\pi}{6}$$

STEP II 2002 Q2

$$z^4 + 5z^3 + 4z^2 - 5z + 1 = 0$$
$$z^2 + 5z + 4 - \frac{5}{z} + \frac{1}{z^2} = 0$$
$$\left(z^2 + \frac{1}{z^2}\right) + 5\left(z - \frac{1}{z}\right) + 4 = 0$$

Setting $w = z + \frac{1}{z}$,

$$w^2 + 2 + 5w + 4 = 0$$
$$w^2 + 5w + 6 = 0$$
$$w = -2, -3$$

So $z - \frac{1}{z} = -2$
$$z^2 + 2z - 1 = 0$$
$$z = \frac{-2 \pm \sqrt{8}}{2}$$

$$z = -1 \pm \sqrt{2}$$

$z - \frac{1}{z} = -3$
$$z^2 + 3z - 1 = 0$$
$$z = \frac{-3 \pm \sqrt{13}}{2}$$

Using the same technique as before,

$$2(z^4 + z^{-4}) - 3(z^3 - z^{-3}) - 12(z^2 + z^{-2}) + 12(z - z^{-1}) + 22 = 0 \qquad (*)$$

And with

$$w^4 = z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$$
$$w^3 = z^3 - 3z + 3z^{-1} - z^{-3}$$
$$w^2 = z^2 - 2 + z^{-2}$$

So $(*)$ becomes

$$\left[2w^4 + 8(z^2 + z^{-2}) - 12\right] + \left[-3w^3 - 9(z - z^{-1})\right] - 12(z^2 + z^{-2}) + 12(z - z^{-1}) + 22$$

$$\Rightarrow 2w^4 - 3w^3 - 4(z^2 + z^{-2}) + 3(z - z^{-1}) + 10 = 0$$
$$\Rightarrow 2w^4 - 3w^3 - 4w^2 - 8 + 3w + 10 = 0$$
$$\Rightarrow 2\left(w^2 + \frac{1}{w^2}\right) - 3\left(w - \frac{1}{w}\right) - 4 = 0$$

Setting $u = w - \frac{1}{w}$,

$$2u^2 + 4 - 3u - 4 = 0$$

$$\Rightarrow \quad 2u^2 - 3u = 0$$

$$\Rightarrow \quad u = 0 \text{ or } 3/2$$

$$w - \frac{1}{w} = 0 \qquad\qquad w - \frac{1}{w} = 3/2$$

$$w^2 - 1 = 0 \qquad\qquad w^2 - \frac{3}{2}w - 1 = 0$$

$$w = \pm 1 \qquad\qquad 2w^2 - 3w - 2 = 0$$

$$(2w+1)(w-2) = 0$$

$$w = -\tfrac{1}{2} \text{ or } 2$$

$$z - \frac{1}{z} = 1 \qquad z - \frac{1}{z} = -1 \qquad z - \frac{1}{z} = 2 \qquad z - \frac{1}{z} = -\frac{1}{2}$$

$$z^2 - z - 1 = 0 \qquad z^2 + z - 1 = 0 \qquad z^2 - 2z - 1 = 0 \qquad 2z^2 + z - 2 = 0$$

$$z = \frac{1 \pm \sqrt{5}}{2} \qquad z = \frac{-1 \pm \sqrt{5}}{2} \qquad z = \frac{2 \pm \sqrt{8}}{2} \qquad z = \frac{-1 \pm \sqrt{17}}{4}$$

$$= 1 \pm \sqrt{2}$$

STEP II 2002 Q3

$F_0 = 2^1 + 1 = 3$
$F_1 = 2^2 + 1 = 5$
$F_2 = 2^4 + 1 = 17$
$F_3 = 2^8 + 1 = 257$

$F_0 = 3 = F_1 - 2$
$F_0 F_1 = 15 = F_2 - 2$
$F_0 F_1 F_2 = 3 \times 5 \times 17 = 15 \times 17$
$= 170 + 85$
$= 255 = F_3 - 2$

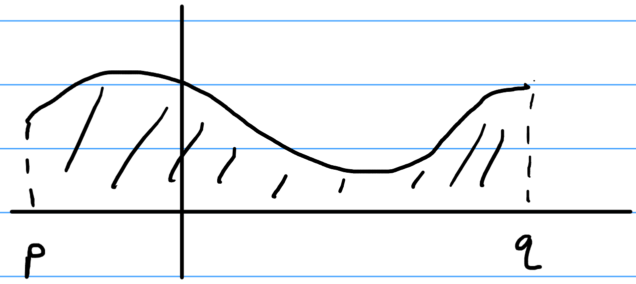
(*) is true for $n=1$. Assume true for $n=k$. Then for $n=k+1$,

$$F_0 F_1 \ldots F_k = F_0 F_1 \ldots F_{k-1} F_k$$
$$= (F_k - 2) F_k \quad \text{by induction assumption}$$
$$= (2^{2^k} - 1)(2^{2^k} + 1)$$
$$= (2^{2^k})^2 - 1$$
$$= 2^{2^k \times 2} - 1$$
$$= 2^{2^{k+1}} - 1$$
$$= F_{k+1} - 2$$, so true for $n \in \mathbb{N}$ by induction.

Suppose $x > 1$ is a factor of F_a, and $a > b$. $x \neq 2$ as F_a is clearly odd.
Then $x | F_a \Rightarrow x \nmid (F_a - 2) \Rightarrow x \nmid (F_0 \ldots F_b)$
$\Rightarrow x \nmid F_b$
So F_a & F_b have no common factors greater than 1.

There are infinitely many Fermat numbers, each with prime factors distinct from each other. So there are infinitely many prime numbers

STEP II 2002 Q4



Area $= \int_p^q f(x)\,dx$ clearly > 0

(i) $ax^2 - bx + c = 0$

$x = b \pm \sqrt{b^2 - 4ac}$

$b^2 < 4ac \Rightarrow$ graph never crosses axis

$a > 0 \Rightarrow$ graph above axis

So combined, $f(x) = ax^2 - bx + c > 0$ for $x \in \mathbb{R}$.

Then for $p < q$, $\int_p^q ax^2 - bx + c \, dx > 0$

$\Rightarrow \left[\frac{ax^3}{3} - \frac{bx^2}{2} + cx\right]_p^q > 0$

$\Rightarrow \frac{a}{3}(q^3 - p^3) - \frac{b}{2}(q^2 - p^2) + c(q-p) > 0$

$\Rightarrow 3b(q^2 - p^2) < 2a(q^3 - p^3) + 6c(q-p)$

Setting $q = 1$, $p = 0$ yields the result.

(ii) $f(x) = a\sin^2 x - b\sin x + c$.

From above, $b^2 < 4ac \Rightarrow f(x) \neq 0$

$f(0) = c > \frac{b^2}{4a} > 0$ as $a > 0$, so $f(x) > 0$ for all x.

So, $\int_p^q a\sin^2 x - b\sin x + c\, dx > 0$

$\Rightarrow \int_p^q \frac{a}{2}(1 - \cos 2x) - b\sin x + c \, dx > 0$

$$\Rightarrow \left[\frac{a}{2}x - \frac{a}{4}\sin 2x + b\cos x + cx\right]_p^q > 0$$

Setting $q = \pi/2$, $p = 0$ gets

$$\left(a\frac{\pi}{4} - 0 + 0 + c\frac{\pi}{2}\right) - (0 + 0 + b + 0) > 0$$

$$\Rightarrow 4b < \pi(a + 2c)$$

(iii) Let $f(x) = \dfrac{a}{x^2} - \dfrac{b}{x} + c$

$$= ax^{-2} - bx^{-1} + c$$

Then $b^2 < 4ac \Rightarrow f(x) \neq 0$ (as quadratic in x^{-1})

$\lim_{x \to \infty} f(x) = c > \frac{b^2}{4a} > 0$, so $f(x) > 0$ for $x > 0$.

Hence $\displaystyle\int_p^q ax^{-2} - bx^{-1} + c \, dx > 0$ for $q > p > 0$

$$\Rightarrow \left[-\frac{a}{x} - b\ln x + cx\right]_p^q > 0$$

$$\Rightarrow a\left(\frac{1}{p} - \frac{1}{q}\right) - b\ln\frac{q}{p} + c(q - p) > 0$$

The result follows.

STEP II 2002 Q5

$x_{n+1} = kx_n(1-x_n)$

(i) $x_{n+1} = kx_n - kx_n^2$
$= k\left[-(x_n-\tfrac{1}{2})^2 + \tfrac{1}{4}\right]$

For $0<x<1$, $0 < -(x-\tfrac{1}{2})^2 + \tfrac{1}{4} < \tfrac{1}{4}$
$\Rightarrow 0 < 4\left[-(x-\tfrac{1}{2})^2 + \tfrac{1}{4}\right] < 1$

So for $0<k<4$ and $0<x_0<1$, we have $0<x_n<1$ for all n.

(ii) $a = ka(1-a)$
$\Rightarrow ka^2 + (1-k)a = 0$
$\Rightarrow ka + (1-k) = 0$
$\Rightarrow k = \frac{1}{1-a}$

(iii) $x_1 = ka(1-a)$
$= ab$

$x_2 = kab(1-ab)$
$= kab - ka^2b^2$
$= \frac{ab^2}{1-a} - \frac{a^2b^3}{1-a}$

So $x_2 = a \Rightarrow a(1-a) = ab^2 - a^2b^3$
$\Rightarrow ab^3 - b^2 + (1-a) = 0$

One solution is $k = \frac{1}{1-a}$
$\Rightarrow b = 1$

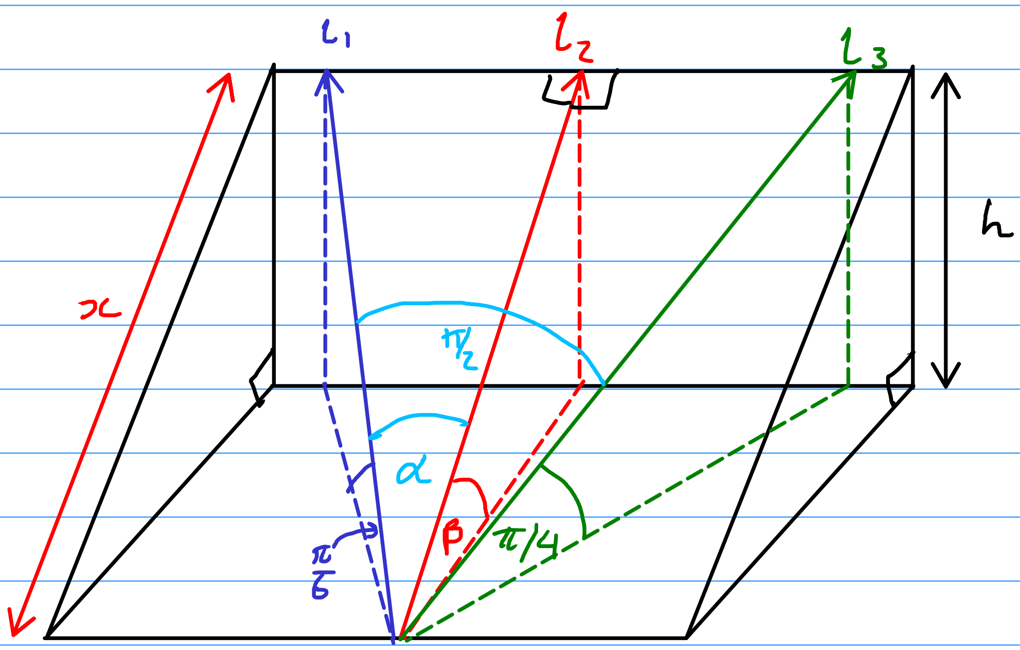
So $(b-1)(ab^2 + (a-1)b + a-1) = 0$

If $k \neq \frac{1}{1-a}$, so $b \neq 1$, $b = \dfrac{1-a \pm\sqrt{(a-1)^2 - 4a(a-1)}}{2a}$

$= \dfrac{1-a\pm\sqrt{(1-a)(1+3a)}}{2a}$

So $k = \frac{b}{1-a} = \frac{1}{2a} \pm \sqrt{\frac{1+3a}{1-a}}$

STEP II 2002 Q6



We have $x\sin\beta = h$, and also $\frac{x}{\cos\alpha}\sin\frac{\pi}{6} = h$

$$\Rightarrow x\sin\beta = \frac{x}{\cos\alpha}\sin\frac{\pi}{6}$$
$$\Rightarrow \cos\alpha\sin\beta = \frac{1}{2}$$

We also have $\frac{x}{\cos(\pi/2-\alpha)}\cdot\sin\frac{\pi}{4} = h$

$$\Rightarrow x\sin\beta = \frac{x}{\sin\alpha}\cdot\sin\frac{\pi}{4}$$
$$\Rightarrow \sin\alpha\sin\beta = \frac{\sqrt{2}}{2}$$

Squaring and adding, $\sin^2\beta(\cos^2\alpha+\sin^2\alpha) = \frac{1}{4}+\frac{1}{2}$

$$\Rightarrow \sin^2\beta = \frac{3}{4}$$
$$\Rightarrow \sin\beta = \sqrt{3}/2 \qquad (\text{as } 0<\beta<\pi/2)$$
$$\Rightarrow \beta = \pi/3$$

Now, using a similar method to before, we have

$$x\sin\beta = \frac{x}{\cos\theta} \cdot \sin\phi$$
$$\Rightarrow \frac{\sqrt{3}}{2}\cos\theta = \sin\phi \qquad (1)$$

And

$$x\sin\beta = \frac{x}{\sin\theta} \cdot \sin 2\phi$$
$$\Rightarrow \frac{\sqrt{3}}{2}\sin\theta = \sin 2\phi \qquad (2)$$

$$\Rightarrow \frac{3}{4}(\cos^2\theta + \sin^2\theta) = \sin^2\phi + \sin^2 2\phi$$
$$\Rightarrow \sin^2\phi + 4\sin^2\phi\cos^2\phi = \frac{3}{4}$$
$$\Rightarrow \sin^2\phi + 4\sin^2\phi(1-\sin^2\phi) = \frac{3}{4}$$
$$\Rightarrow 4\sin^4\phi - 5\sin^2\phi + \frac{3}{4} = 0$$

$$\Rightarrow \sin^2\phi = \frac{5 \pm \sqrt{25-12}}{8}$$

$$= \frac{5 \pm \sqrt{13}}{8}$$

Going back and dividing (2) by (1), we have

$$\tan\theta = \frac{\sin 2\phi}{\sin\phi}$$

$$\Rightarrow \tan^2\theta = \frac{\sin^2 2\phi}{\sin^2\phi}$$

$$= 4\cos^2\phi$$
$$= 4\left(1 - \frac{5 \pm \sqrt{13}}{}\right)$$
$$= \frac{3 \mp \sqrt{13}}{8} \quad \text{but } \tan^2\theta > 0, \text{ so}$$

$$\tan\theta = \frac{3 + \sqrt{13}}{8}$$

STEP II 2002 Q7

We have $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$. Suppose another line has direction vector $\begin{pmatrix}1\\b\\c\end{pmatrix}$. If it makes an angle $\pi/4$ with the previous lines, we have

$$\begin{pmatrix}1\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = \sqrt{1+b^2+c^2}\cdot\sqrt{1^2+1^2}\cdot\frac{1}{\sqrt{2}}$$

$\Rightarrow 1+b = \sqrt{1+b^2+c^2}$

$\Rightarrow b^2+2b+1 = 1+b^2+c^2$

$\Rightarrow 2b = c^2$

Similarly from $\begin{pmatrix}1\\0\\1\end{pmatrix}$ we get $2c = b^2$

Hence $2b = c^2 = \left(\frac{b^2}{2}\right)^2 = \frac{b^4}{4}$

$\Rightarrow 8b - b^4 = 0$

$\Rightarrow b(8-b^3) = 0$

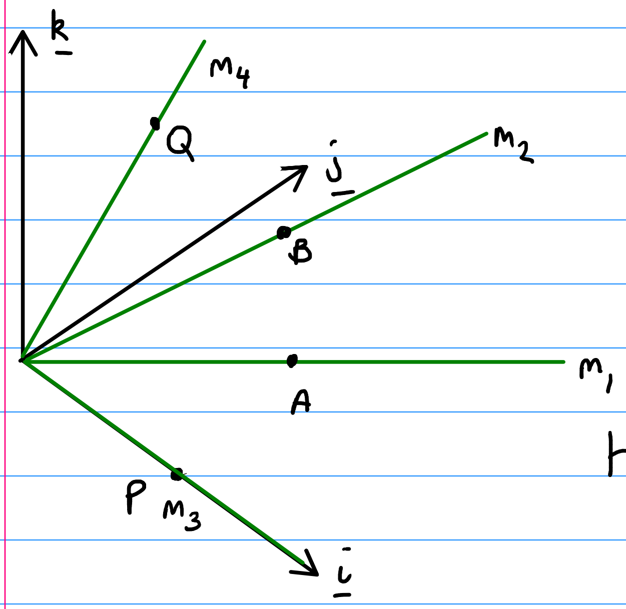
$\Rightarrow b = 0$ or 2

$c = 0$ or 2

So m_3 and m_4 have direction vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\2\\2\end{pmatrix}$. If θ is the acute angle between these, then

$$\cos\theta = \frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}}{1\cdot\sqrt{9}}$$

$$= \frac{1}{3}$$



we have

$$A = \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} \quad B = \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$$
$$P = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad Q = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

Hence $AQ = \begin{pmatrix}\frac{1}{3}-\lambda\\ \frac{2}{3}-\lambda\\ 2/3\end{pmatrix}$

$$BP = \begin{pmatrix}1-\lambda\\0\\-\lambda\end{pmatrix}$$

If AQ is perpendicular to BP, then $AQ \cdot BP = 0$

$$\Rightarrow \left(\tfrac{1}{3}-\lambda\right)(1-\lambda) - \tfrac{2}{3}\lambda = 0$$
$$\Rightarrow \lambda^2 - \tfrac{4}{3}\lambda + \tfrac{1}{3} - \tfrac{2}{3}\lambda = 0$$
$$\Rightarrow 3\lambda^2 - 6\lambda + 1 = 0$$

$\Delta = 36 - 12$

$= 24 > 0$ so there are exactly two distinct solutions for λ.

A general point on the line AQ is

$$\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix} + \alpha\begin{pmatrix}1/3-\lambda\\2/3-\lambda\\2/3\end{pmatrix}$$

and on BP is

$$\begin{pmatrix}1\\0\\0\end{pmatrix} + \rho\begin{pmatrix}1-\lambda\\0\\-\lambda\end{pmatrix}$$

Solving these simultaneously to find a point of intersection,

$$\frac{1}{3}+\frac{1}{3}\alpha-\alpha\lambda=1+\beta-\beta\lambda \qquad (1)$$

$$\frac{2}{3}+\frac{2}{3}\alpha-\alpha\lambda=0 \qquad (2)$$

$$\frac{2}{3}+\frac{2}{3}\alpha=-\beta\lambda \qquad (3)$$

(2) $\Rightarrow \quad 2+2\alpha-3\alpha\lambda=0$

$\Rightarrow \quad \alpha=\dfrac{-2}{2-3\lambda}$ so $\lambda\neq\frac{2}{3}$ (then (2) is $\frac{2}{3}=0$ ※)

Substituting into (3)

$$2-\frac{4}{2-3\lambda}=-3\beta\lambda$$

$$\Rightarrow -3\beta\lambda=\frac{-6\lambda}{2-3\lambda} \qquad (\lambda\neq 0)$$

$$\Rightarrow \beta=\frac{2}{2-3\lambda}$$

Substituting into $3\times$(1),

$$1-\frac{2}{2-3\lambda}+\frac{6\lambda}{2-3\lambda}=3+\frac{6}{2-3\lambda}-\frac{6\lambda}{2-3\lambda}$$

$\Rightarrow \quad 12\lambda-8=2(2-3\lambda)$

$\Rightarrow \quad 12\lambda-8=4-6\lambda$

$\Rightarrow \quad 18\lambda=12$

$\Rightarrow \quad \lambda=\frac{2}{3}$ ※ as we know $\lambda\neq\frac{2}{3}$.

So there are no non-zero values of λ for which AQ and BP intersect.

STEP II 2002 Q8

For $x<0$, $\frac{dy}{dx} = -y \Rightarrow y = Ae^{-x}$

$y(-1) = a \Rightarrow a = Ae$

$\Rightarrow A = a/e$

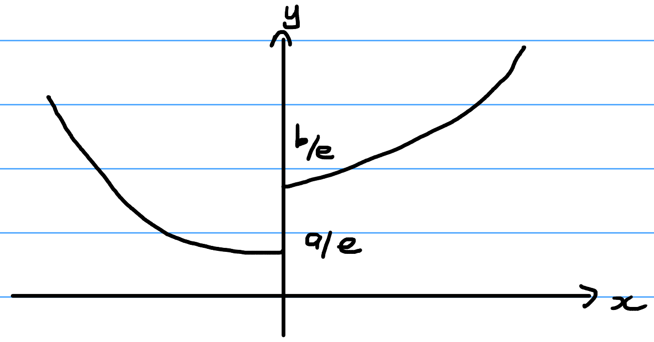
$\Rightarrow y = ae^{-x-1}$

For $x>0$, $\frac{dy}{dx} = y \Rightarrow y = Be^{x}$

$y(1) = b \Rightarrow b = Be$

$\Rightarrow B = b/e$

$\Rightarrow y = be^{x-1}$



For continuity, $\frac{a}{e} = \frac{b}{e} \Rightarrow a = b$

For $x>0$, $\frac{dy}{dx} = (e^{x}-1)y$

$\Rightarrow \frac{1}{y}dy = (e^{x}-1)dx$

$\Rightarrow \ln y = e^{x} - x + c$

$\Rightarrow y = e^{e^{x}-x+c}$

$y(1) = e^{e} \Rightarrow e - 1 + c = e$

$\Rightarrow c = 1$

So $y = e^{e^{x}-x+1}$

Then $y(0) = e^{2}$

For $x<0$, $\frac{dy}{dx} = (1-e^{x})y$

$\Rightarrow \frac{1}{y}dy = (1-e^{x})dx$

$\Rightarrow \ln y = x - e^{x} + c$

$\Rightarrow y = e^{x-e^{x}+c}$

$y(0) = e^{2} \Rightarrow 0 - 1 + c = 2$

$\Rightarrow c = 3$

So $y = e^{x-e^{x}+3}$

(i) $\lim_{x\to\infty} ye^{-e^{x}}$

$= \lim_{x\to\infty} e^{e^{x}-x+1}e^{-e^{x}}$

$= \lim_{x\to\infty} e^{-x+1}$

$= 0$

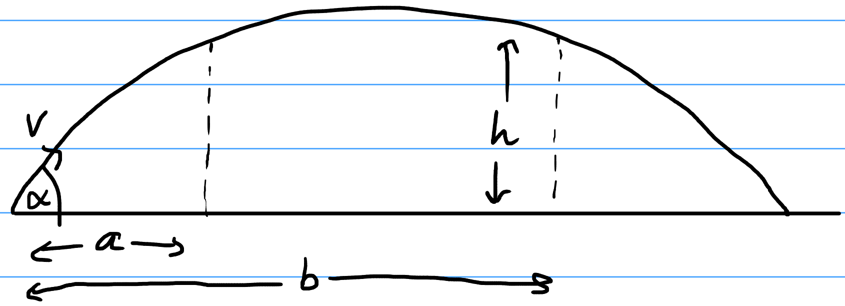
(ii) $\lim_{x\to-\infty} ye^{-x}$

$= \lim_{x\to-\infty} e^{x-e^{x}+3}e^{-x}$

$= \lim_{x\to-\infty} e^{-e^{x}+3}$

$= e^{3}$

STEP II 2002 Q9



s	h
u	$v\sin\alpha$
v	x
a	$-g$
t	$\frac{a}{v\cos\alpha}$

$$\Rightarrow h = v\sin\alpha \cdot \frac{a}{v\cos\alpha} - \frac{1}{2}g\frac{a^2}{v^2\cos^2\alpha}$$

$$= a\tan\alpha - \frac{a^2 g}{2v^2\cos^2\alpha} \quad (1)$$

Similarly,
$$h = b\tan\alpha - \frac{b^2 g}{2v^2\cos^2\alpha}$$

$$\Rightarrow \frac{g}{2v^2} = \frac{b\tan\alpha - h}{b^2} \cdot \cos^2\alpha$$

Substituting into (1),

$$h = a\tan\alpha - \frac{a^2}{\cos^2\alpha} \cdot \frac{b\tan\alpha - h}{b^2} \cdot \cos^2\alpha$$

$$\Rightarrow b^2 h = ab^2\tan\alpha - a^2 b\tan\alpha + a^2 h$$

$$\Rightarrow h(b^2 - a^2) = ab\tan\alpha(b-a)$$

$$\Rightarrow \tan\alpha = \frac{h(a+b)}{ab}$$

Hence $\dfrac{2V^2}{g} = \dfrac{b^2}{\cos^2\alpha\,(b\tan\alpha - h)}$

$$= \frac{b^2}{b\tan\alpha - h}\cdot(1+\tan^2\alpha)$$

$$= \frac{b^2}{h\left(\frac{a+b}{a} - 1\right)}\cdot\left(1 + \frac{h^2(a+b)^2}{a^2b^2}\right)$$

$$= \frac{ab}{h}\left(1 + \frac{h^2(a+b)^2}{a^2b^2}\right)$$

$$= \frac{ab}{h} + \frac{h(a+b)^2}{ab} \qquad (*)$$

We have $\tan\alpha = \dfrac{h(a+b)}{ab}$

$$\Rightarrow \sec^2\alpha\frac{d\alpha}{dh} = \frac{a+b}{ab}$$

$$\Rightarrow \sec^2\alpha\,\delta a \approx \frac{a+b}{ab}\delta h$$

$\sec^2\alpha$, $\frac{a+b}{ab}$, δh are all +ve, so δa is +ve.

From $(*)$,

$$\frac{4V}{g}\delta V = \frac{-ab}{h^2}\delta h + \frac{(a+b)^2}{ab}\delta h$$

$$= \delta h\left[\frac{h^2(a+b)^2 - a^2b^2}{h^2ab}\right]$$

So $\delta V > 0 \Leftrightarrow h^2(a+b)^2 - a^2b^2 > 0$

$$\Leftrightarrow (h(a+b) + ab)(h(a+b) - ab) > 0$$

$$\Leftrightarrow h(a+b) - ab > 0$$

$$\Leftrightarrow h > \frac{ab}{a+b}$$

STEP II 2002 Q10

$$42\tfrac{3}{8} = 13t + \left(14 + \frac{2t}{T}\right)(T-t)$$

$$\Rightarrow 42\tfrac{3}{8}T = 13Tt + 14T^2 - 14Tt + 2Tt - 2t^2$$

$$\Rightarrow 0 = -42\tfrac{3}{8}T + 14T^2 + Tt - 2t^2$$

$$\Rightarrow 0 = -42\tfrac{3}{8}\frac{dT}{dt} + 28T\frac{dT}{dt} + t\frac{dT}{dt} + T - 4t \quad (*)$$

$$\frac{dT}{dt} = 0 \Rightarrow T = 4t$$

Hence $42\tfrac{3}{8} = 13t + \left(14 + \tfrac{1}{2}\right)(3t)$

$$\Rightarrow 42\tfrac{3}{8} = 13t + \frac{87}{2}t$$

$$\Rightarrow \frac{339}{8} = \frac{113}{2}t$$

$$\Rightarrow t = \frac{3}{4} \Rightarrow T = 3$$

differentiating $(*)$ again,

$$0 = -42\tfrac{3}{8}\frac{d^2T}{dt^2} + 28\left(\cancel{\frac{dT}{dt}}\right)^2 + 28T\frac{d^2T}{dt^2} + t\frac{d^2T}{dt^2} + \cancel{\frac{dT}{dt}} + \cancel{\frac{dT}{dt}} - 4$$

$$\Rightarrow 4 = \frac{d^2T}{dt^2}\left(-42\tfrac{3}{8} + 112t + t\right)$$

$$\Rightarrow \frac{d^2T}{dt^2} = \frac{4}{113t - 42\tfrac{3}{8}}$$

$t = \frac{3}{4} \Rightarrow \frac{d^2T}{dt^2} = \frac{4}{84\tfrac{3}{4} - 42\tfrac{3}{8}} > 0$ so minimum.

For the first competitor for the first $\frac{3}{4}$ hour, $s_1(t) = 13t$

After that, $s_1(t) = 13 \cdot \frac{3}{4} + \frac{29}{2}(t - \frac{3}{4})$

$$= \frac{29}{2}t - \frac{9}{8}$$

For the second competitor, $s = ut + \frac{1}{2}at^2$

$$\Rightarrow 42\tfrac{3}{8} = 48 + \frac{9}{2}a$$

$$\Rightarrow -\frac{45}{8} = \frac{9}{2}a$$

$$\Rightarrow a = \frac{-5}{4}$$

So $s_2(t) = 16t - \frac{5}{8}t^2$

For $0 < t < \frac{3}{4}$, difference is $16t - \frac{5}{8}t^2 - 13t$

$$= 3t - \frac{5}{8}t^2$$

$$= t(3 - \frac{5}{8}t)$$

maximised at $t = \frac{12}{5}$ not in range, so max at $t = \frac{3}{4}$ $= \frac{3}{4}(3 - \frac{5}{8} \cdot \frac{3}{4})$

$$= \frac{3}{4}(3 - \frac{15}{32})$$

$$= \frac{3}{4} \cdot \frac{81}{32} = \frac{243}{128}$$

For $\frac{3}{4} < t < 3$, difference is $16t - \frac{5}{8}t^2 - \frac{29}{2}t + \frac{9}{8}$

$$= -\frac{5}{8}t^2 + \frac{3}{2}t + \frac{9}{8}$$

$\frac{d}{dt}$: $-\frac{5}{4}t + \frac{3}{2} = 0$

$$\Rightarrow t = \frac{3}{2} \cdot \frac{4}{5}$$

$$= \frac{6}{5}$$

Then difference is $-\frac{5}{8} \cdot \frac{36}{25} + \frac{3}{2} \cdot \frac{6}{5} + \frac{9}{8}$

$$= \frac{-9}{10} + \frac{9}{5} + \frac{9}{8}$$

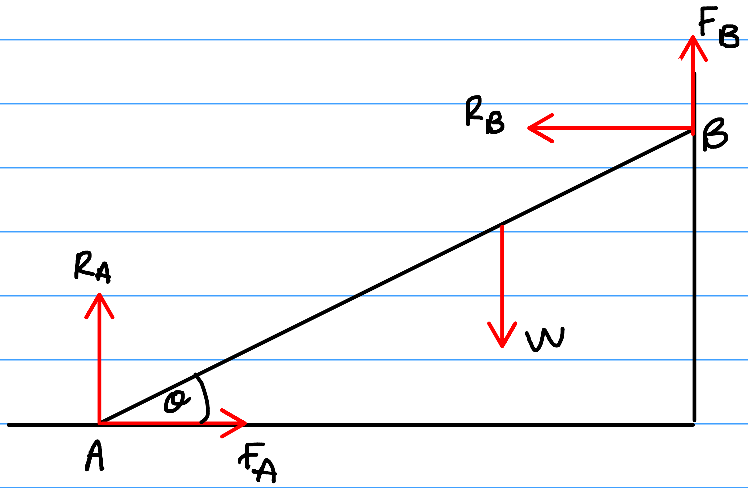
$= \frac{81}{40} > \frac{243}{128}$, so maximum distance is $\frac{81}{40}$km.

STEP II 2002 Q11

$$\text{CoM} = \frac{\int_0^l \alpha \cancel{W} l^{-1}\left(\frac{x}{l}\right)^{\alpha-1} x\,dx}{\cancel{W}}$$

$$= \frac{\alpha}{l^\alpha}\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^l$$

$$= \frac{\alpha \cdot l^{\alpha+1}}{l^\alpha(\alpha+1)} = \frac{\alpha l}{\alpha+1}$$



$R(\leftrightarrow)$: $F_A = R_B$ ① $F_A = \mu R_A$ ③

$R(\updownarrow)$: $R_A + F_B = W$ ② $F_B = \mu R_B$ ④

$M(A)$: $W \cdot \frac{l}{\alpha+1}\cos\theta = R_B l\sin\theta + F_B l\cos\theta$ ⑤

① & ③ $\Rightarrow \mu R_A = R_B$

② & ④ $\Rightarrow R_A + \mu R_B = W$

$\Rightarrow R_A + \mu^2 R_A = W$

$\Rightarrow W = (1+\mu^2)R_A$

④ & ⑤ $\Rightarrow$

$$\frac{W}{\alpha+1}\cos\theta = \mu R_A \sin\theta + \mu^2 R_A\cos\theta$$

Combining these, $\frac{(1+\mu^2)\cancel{R_A}}{\alpha+1}\cos\theta = \mu\cancel{R_A}(\sin\theta + \mu\cos\theta)$

$$\Rightarrow (1+\mu^2)\cos\theta = \mu(\alpha+1)(\sin\theta+\mu\cos\theta)$$

$$\Rightarrow \cos\theta(1+\cancel{\mu^2}-\alpha\mu^2-\cancel{\mu^2}) = \mu(\alpha+1)\sin\theta$$

$$\Rightarrow \tan\alpha = \frac{1-\alpha\mu^2}{(1+\alpha)\mu}$$

Now $\alpha = \frac{3}{2}$, so CoM's $l \cdot \frac{3/2}{5/2} = 3/5\, l$ from B

$\tan \pi/4 = 1$ so $\frac{5}{2}\mu = 1 - \frac{3}{2}\mu^2$

$$\Rightarrow 5\mu = 2 - 3\mu^2$$
$$\Rightarrow 3\mu^2 + 5\mu - 2 = 0$$
$$\Rightarrow (3\mu-1)(\mu+2)$$
$$\Rightarrow \mu = 1/3, \cancel{-2}$$

So the maximum value of μ is $1/3$.

STEP II 2002 Q12

Number of heads from a particular coin $\sim B(M, p)$

$$\approx N(Mp, Mp(1-p))$$

So $P(\text{particular coin has} < m \text{ heads}) \approx \Phi\left(\frac{m-\frac{1}{2}-Mp}{\sqrt{Mp(1-p)}}\right)$

$$= \Phi\left(\frac{2m-1-2Mp}{\sqrt{Mp(1-p)}}\right)$$

So number of coins with $< m$ heads $\sim B\left(L, \Phi\left(\frac{2m-1-2Mp}{\sqrt{Mp(1-p)}}\right)\right)$

$$\approx N(h, h(1-\tfrac{h}{L}))$$

So $P(\text{more than } l \text{ coins have} < m \text{ heads}) \approx P\left(Z > \frac{l+\frac{1}{2}-h}{\sqrt{h(1-h/L)}}\right)$

$$= P\left(Z < \frac{2h-2l-1}{2\sqrt{h(1-h/L)}}\right)$$

$$\approx P\left(Z < \frac{2h-2l-1}{2\sqrt{h}}\right)$$

assuming h/L is small

Then number of days with $> l$ coins with $< m$ heads $\sim B(K, q)$,

so $P(k \text{ days with} > l \text{ coins with} < m \text{ heads}) = \binom{K}{k} q^k (1-q)^{K-k}$

With the given numbers the first approximation is $B(100, 0.6)$ which is well approximated by a normal.

Then $\Phi\left(\frac{2m-1-2Mp}{\sqrt{Mp(1-p)}}\right) = \Phi\left(\frac{96-1-120}{2\sqrt{24}}\right)$

$$= \Phi\left(\frac{-25}{4\sqrt{6}}\right)$$

$$= \Phi\left(-\frac{25}{24}\sqrt{6}\right)$$

$$\approx \Phi(-2.5)$$

$$\approx 0.01 \qquad (\Phi(-2) \approx 0.025, \Phi(3) \approx 0.0015)$$

Then the second approximation is to $B(500, 0.01)$, which has $\mu = np = 5$, which is small, so not a good approximation.

STEP II 2002 Q13

$$G(y) = \frac{F(y)}{2-F(y)}, \quad \text{then} \quad G(a) = \frac{F(a)}{2-F(a)}$$

$$= \frac{0}{2}$$

$$= 0$$

$$G(1) = \frac{F(1)}{2-F(1)}$$

$$= \frac{1}{2-1}$$

$$= 1$$

$$G'(y) = \frac{F'(y)(2-F(y)) + F'(y)F(y)}{(2-F(y))^2}$$

$$= \frac{2F'(y)}{(2-F(y))^2} \geqslant 0 \quad \text{because } F'(y) \geqslant 0.$$

We have $0 \leq F(y) \leq 1$

$$\Rightarrow \quad 1 \leq 2 - F(y) \leq 2$$

$$\Rightarrow \quad 1 \leq [(2-F(y)]^2 \leq 4$$

$$\Rightarrow \quad \frac{1}{2} \leq \frac{2}{(2-F(y))^2} \leq 2$$

Hence $\frac{1}{2}F'(y) \leq G'(y) \leq 2F'(y)$

$$\Rightarrow \quad \frac{1}{2}\int_a^b yF'(y)\,dy \leq \int_a^b yG'(y)\,dy \leq 2\int_a^b yF'(y)$$

because $a, b > 0$ so $y > 0$.

$\Rightarrow \quad \frac{1}{2} EX \leq EY \leq 2EX$

Using a similar argument but with y^2 rather than y,

$$EY^2 \leq 2EX^2$$
$$= 2\,\mathrm{Var}X + 2(EX)^2$$

So $\mathrm{Var}Y = EY^2 - (EY)^2$
$$\leq 2\,\mathrm{Var}X + 2(EX)^2 - \frac{1}{4}(EX)^2$$
$$= 2\,\mathrm{Var}X + \frac{7}{4}(E(X))^2$$

STEP II 2002 Q14

The area of region $n \propto n^2-(n-1)^2$
$= 2n-1$

So $P(\text{region } n) = \frac{2n-1}{N^2}$

Hence long term average is

$$\sum_{n=1}^{N} \frac{2n-1}{N^2}\left(2-\frac{n}{N}\right)$$

$$= \frac{1}{N^2}\sum_{n=1}^{N} 4n-2-\frac{2n^2}{N}+\frac{n}{N}$$

$$= \frac{1}{N^2}\left[-2N+\left(4+\frac{1}{N}\right)\sum_{n=1}^{N} n - \frac{2}{N}\sum_{n=1}^{N} n^2\right]$$

$$= \frac{1}{N^{\cancel{2}}}\left[-2\cancel{N}+\left(4+\frac{1}{N}\right)\frac{1}{2}\cancel{N}(N+1)-\frac{2}{N}\cdot\frac{1}{6}\cancel{N}(N+1)(2N+1)\right]$$

$$= \frac{1}{N}\left[-2+\frac{1}{2}(N+1)\left(4+\frac{1}{N}\right)-\frac{1}{3N}(N+1)(2N+1)\right]$$

$$= \frac{1}{6N^2}\left[-12N+3(N+1)(4N+1)-2(N+1)(2N+1)\right]$$

$$= \frac{1}{6N^2}\left[-12N+12N^2+15N+3-4N^2-6N-2\right]$$

$$= \frac{1}{6N^2}\left[8N^2-3N+1\right]$$

$$= \frac{1}{6N^2}\left[12N^2-4N^2-3N+1\right]$$

$$= 2-\frac{1}{6}\left(4+\frac{3}{N}-\frac{1}{N^2}\right)$$

$$= 2-\frac{1}{6}\left(1+\frac{1}{N}\right)\left(4-\frac{1}{N}\right), \text{ as required.}$$

$$P(X=k) = \sum_{n=1}^{N} \frac{2n-1}{N^2} \cdot e^{-2+\frac{n}{N}} \frac{\left(2-\frac{n}{N}\right)^k}{k!}$$

$$= \frac{e^{-2}}{k!N^2} \sum_{n=1}^{N} (2n-1) e^{\frac{n}{N}} \frac{1}{N^k} (2N-n)^k$$

$$= \frac{e^{-2}N^{-k-2}}{k!} \sum_{n=1}^{N} (2n-1)(2N-n)^k e^{n/N}$$

For $N=3$, $P(R_2 \text{ selected} \mid X=2)$

$$= \frac{P(R_2 \cap X=2)}{P(X=2)}$$

$$= \frac{\frac{2\times 2-1}{3^2} e^{-2+\frac{2}{3}} \cdot \frac{\left(2-\frac{2}{3}\right)^2}{2!}}{\frac{e^{-2}3^{-4}}{2!} \sum_{n=1}^{3} (2n-1)(6-n)^2 e^{n/3}}$$

$$= \frac{\frac{1}{3} e^{-4/3} \cdot \frac{8}{9}}{\frac{e^{-2}}{2\times 81}\left(25e^{1/3} + 48e^{2/3} + 45e\right)}$$

$$= \frac{48}{e^{-2/3}\left(25e^{1/3} + 48e^{2/3} + 45e\right)}$$

$$= \frac{48}{48 + 45e^{1/3} + 25e^{-1/3}}$$