STEP I 2002 Q1

$(x+2)^2+2y^2=18$ ① $9(x-1)^2+16y^2=25$ ②

① $\Rightarrow 16y^2 = 144-8(x+2)^2$

Into ② $144-8(x+2)^2 = 25-9(x-1)^2$

$\Rightarrow 144-8x^2-32x-32 = 25-9x^2+18x-9$

$\Rightarrow x^2-50x+96=0$

$\Rightarrow (x-48)(x-2)=0$

$x=48 \Rightarrow 50^2+2y^2=18$

$\Rightarrow y^2 = 9-\frac{1}{2}\times 5^2 < 0$ #

$x=2 \Rightarrow 16+2y^2=18$

$\Rightarrow y=\pm 1$

So the circle passes through $(2,1)$ and $(2,-1)$. Hence the x axis is a line of symmetry for the circle and so its centre lies on the x axis, say at $(a,0)$. Then the radius must be $r^2=(a-2)^2+1^2$

So $(x-a)^2+y^2=(a-2)^2+1$

$\Rightarrow x^2-2ax+a^2+y^2=a^2-4a+5$

$\Rightarrow x^2-2ax+y^2=5-4a$

STEP I 2001 Q2

$$f(x) = x^m (x-1)^n$$

$$f'(x) = x^{m-1}(x-1)^{n-1}\left[m(x-1) + nx\right] = 0$$

$$\Rightarrow \quad x^{m-1}(x-1)^{n-1}\left[(n+m)x - m\right] = 0$$

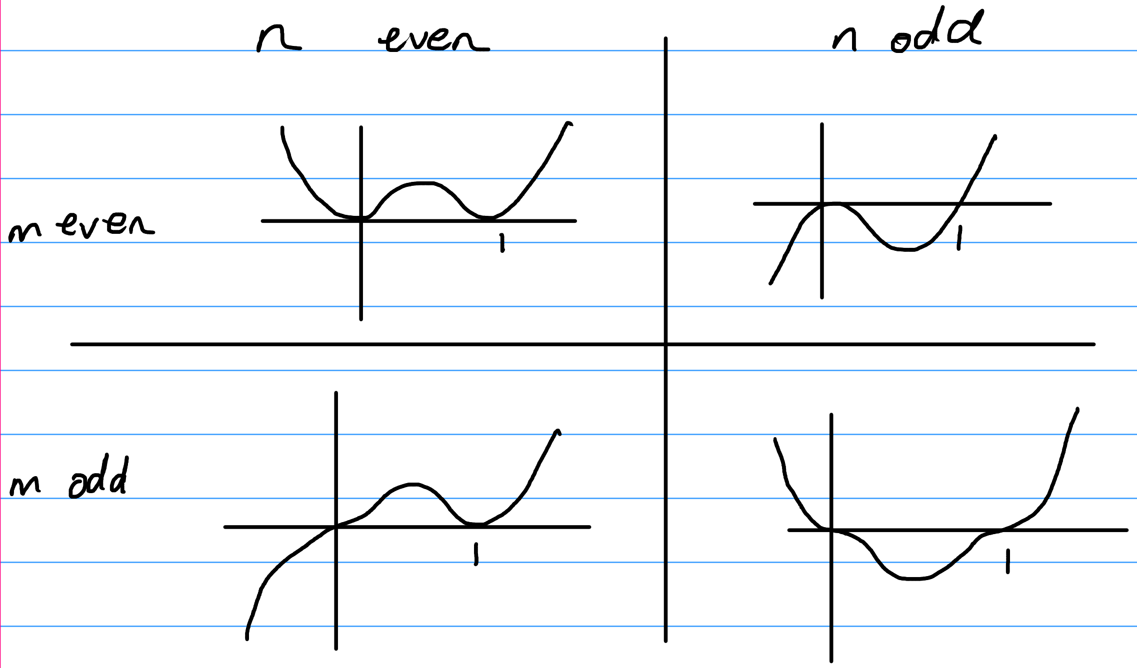
$\Rightarrow$ stationary point at $x = \frac{m}{n+m} \in (0,1)$ for $0 < n, m$.

$$f''(x) = (m-1)x^{m-2}(x-1)^{n-1}\left[(n+m)x - m\right]$$
$$+ (n-1)x^{m-1}(x-1)^{n-2}\left[(n+m)x-m\right]$$
$$+ (n+m)x^{m-1}(x-1)^{n-1}$$
$$= x^{m-2}(x-1)^{n-2}\Big[(m-1)(x-1)\left[(n+m)x - m\right]$$
$$+(n-1)x\left[(n+m)x-m\right]$$
$$+(n+m)x(x-1)\Big]$$

$$\text{So } f''\left(\frac{m}{n+m}\right) = \left(\frac{m}{n+m}\right)^{m-2}\left(\frac{-n}{n+m}\right)^{n-2} \cdot \left[0 + 0 - \frac{mn}{m+n}\right]$$

$$= \frac{(-1)^{n-1} m^{m-1} n^{n-1}}{(n+m)^{m+n-3}}$$

which is < 0 if n even $\Rightarrow$ maximum, and > 0 if n odd $\Rightarrow$ minimum.



STEP I 2001 Q3

$$2a^2+2b^2-(a+b)^2$$
$$= a^2+b^2-2ab$$
$$= (a-b)^2 \geq 0$$
$$y = (a^2\cos^2\theta + b^2\sin^2\theta)^{1/2} + (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}$$

$$\frac{dy}{d\theta} = \left[\frac{(b^2-a^2)\sin\theta\cos\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}} + \frac{(a^2-b^2)\sin\theta\cos\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}}\right] = 0$$

$$\Rightarrow \sin\theta\cos\theta = 0 \text{ or } (b^2-a^2)(a^2\sin^2\theta+b^2\cos^2\theta)+(a^2-b^2)(a^2\cos^2\theta+b^2\sin^2\theta)=0$$

$$\sin\theta\cos\theta = 0 \Rightarrow \sin 2\theta = 0$$
$$\Rightarrow \theta = k\pi,\ (k+\tfrac{1}{2})\pi$$

In both cases, $y = |a| + |b|$

If $(b^2-a^2)(a^2\sin^2\theta+b^2\cos^2\theta)+(a^2-b^2)(a^2\cos^2\theta+b^2\sin^2\theta)=0$, we have

$$a^2\sin^2\theta + b^2\cos^2\theta - a^2\cos^2\theta - b^2\sin^2\theta = 0$$
$$\Rightarrow a^2\sin^2\theta + b^2(1-\sin^2\theta) - a^2(1-\sin^2\theta) - b^2\sin^2\theta = 0$$
$$\Rightarrow \sin^2\theta(2a^2-2b^2) = a^2-b^2$$
$$\Rightarrow \sin^2\theta = \tfrac{1}{2}$$

[note $a=b$ gives $y = 2|a|$ constant]

$$\Rightarrow \theta = (k \pm \tfrac{1}{4})\pi$$
$$\Rightarrow y = 2\sqrt{\frac{a^2+b^2}{2}}$$
$$= \sqrt{2(a^2+b^2)}$$

Because $(a+b)^2 \leq 2a^2+2b^2$ we have $a+b \leq \sqrt{2a^2+2b^2}$

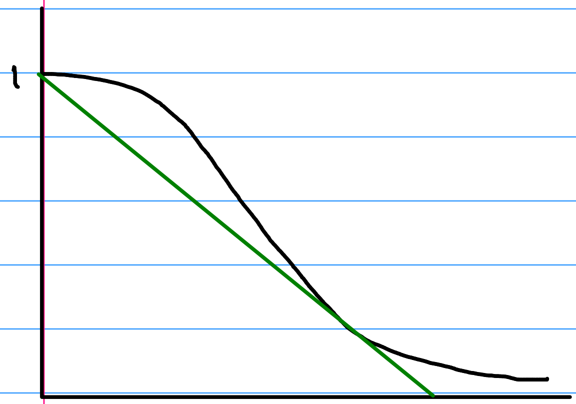
$$\Rightarrow |a| + |b| \leq \sqrt{2a^2+2b^2}$$

as RHS is invariant under $a \to -a$

Hence $\theta = k\pi, (k+\frac{1}{2})\pi$ gives minima and $\theta = (k \pm \frac{1}{4})\pi$ maxima. As the function is continuous, we have

$$|a|+|b| \leq (a^2\cos^2\theta + b^2\sin^2\theta)^{1/2} + (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2} \leq (2a^2+2b^2)^{1/2}$$

STEP I 2002 Q4



Intersects at $x=0 \Rightarrow y = kx+1$.

Supposing it is tangent to the curve at $x=a$, comparing gradients we have

$$k = \frac{-2a}{(1+a^2)^2}$$

So the equation of the tangent is $y = \frac{-2a}{(1+a^2)^2}x + 1$

But we must have

$$\frac{1}{1+a^2} = \frac{-2a}{(1+a^2)^2}a + 1$$

$$\Rightarrow \quad 1+a^2 = -2a^2 + a^4 + 2a^2 + 1$$

$$\Rightarrow \quad a^4 - a^2 = 0$$

$$\Rightarrow \quad a^2(a-1)(a+1) = 0$$

$$\Rightarrow \quad a = 0, 1, \cancel{-1} \quad \text{as } a \geqslant 0$$

So $k = \frac{-2}{(1+1)^2} = -\frac{1}{2}$. So $y = -\frac{1}{2}x + 1$.

Setting $-\frac{1}{2}x + 1 = \frac{1}{1+x^2}$

$$\Rightarrow -\frac{1}{2}x + 1 - \frac{1}{2}x^3 + x^2 = 1$$

$$\Rightarrow -\frac{1}{2}x(x^2 - 2x + 1) = 0$$

$$\Rightarrow \quad x(x-1)^2 = 0$$

So the only solutions are $x=0$ and $x=1$.

For $0 < x < 1$, $-\frac{1}{2}x + 1 < \frac{1}{1+x^2}$

$$\Rightarrow \int_0^1 -\frac{1}{2}x + 1 \, dx < \int_0^1 \frac{1}{1+x^2} \, dx$$

$$\Rightarrow \left[-\frac{1}{4}x^2 + x\right]_0^1 < \frac{\pi}{4}$$

$$\Rightarrow \frac{3}{4} < \frac{\pi}{4}$$

$$\Rightarrow \pi > 3$$

Considering volume around the y axis for $\frac{1}{2} \le y \le 1$, the line gives a cone of volume $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 1^2 \times \frac{1}{2} = \frac{\pi}{6}$.

The volume of the curve rotated is $\int_{1/2}^{1} \pi x^2 dy$

$$= \pi \int_{1/2}^{1} \frac{1}{y} - 1 \, dy$$

$$= \pi \left[\ln y - y\right]_{1/2}^{1}$$

$$= \pi \left[(0 - 1) - (-\ln 2 - \frac{1}{2})\right]$$

$$= \pi\left(\ln 2 - \frac{1}{2}\right)$$

So $\pi\left(\ln 2 - \frac{1}{2}\right) > \frac{\pi}{6}$

$$\Rightarrow \ln 2 - \frac{1}{2} > \frac{1}{6}$$

$$\Rightarrow \ln 2 > \frac{2}{3}$$

STEP I 2002 Q5

$f(x) = x^n + a_1 x^{n-1} + \_\_ + a_n$

$= (x+k_1)(x+k_2)\_\_(x+k_n)$

$f(0) = k_1 \times k_2 \times \_ \times k_n = a_n$

$f(1) = (1+k_1)(1+k_2)\_\_(1+k_n) = 1 + a_1 + a_2 + \_\_ + a_n$

$f(-1) = (k_1-1)(k_2-1)\_\_(k_n-1) = (-1)^n + (-1)^{n-1}a_1 + (-1)^{n-2}a_2 + \_\_ + a_n$

Consider $x^4 + 22x^3 + 172x^2 + 552x + 576$

we have $k_1k_2k_3k_4 = 576$

$(k_1+1)(k_2+1)(k_3+1)(k_4+1) = 1323$

$(k_1-1)(k_2-1)(k_3-1)(k_4-1) = 175$

Note $175 = 5^2 \times 7$

Does $k_1 = 2,\ k_2 = 6,\ k_3 = 6,\ k_4 = 8$ work?

$3 \times 7 \times 7 \times 9 = 1323$ ✓

$2 \times 6 \times 6 \times 8 = 576$

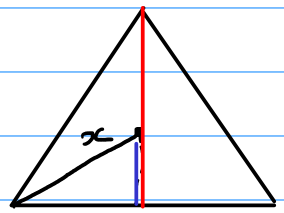
So the roots are $x = -2, -6, -8$

STEP I 2002 Q6

The area of the base is $\frac{1}{2}a^2\sin 60^\circ$

$= \frac{\sqrt{3}}{4}a^2$

Considering the base,



$x = \frac{a}{2} \div \cos 30^\circ$

$= \frac{a}{2} \times \frac{2}{\sqrt{3}}$

$= a\frac{\sqrt{3}}{3}$

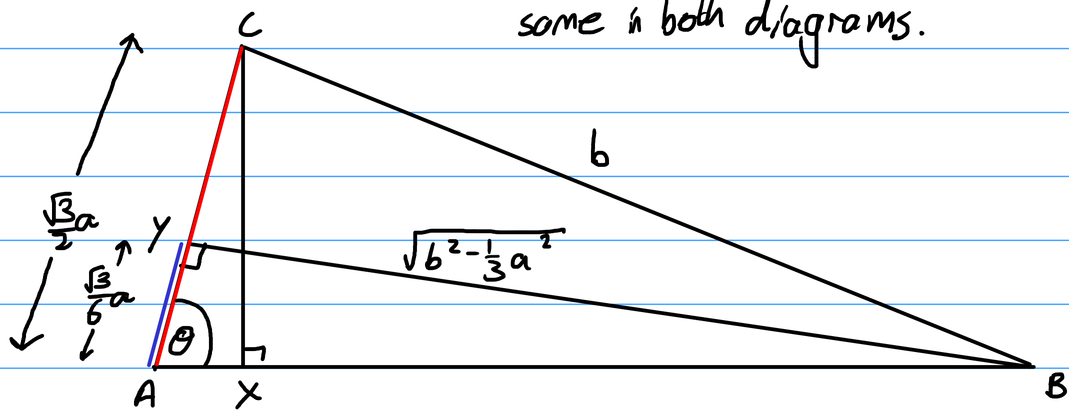
So, the vertical height of the pyramid is $h = \sqrt{b^2 - x^2}$

$= \sqrt{b^2 - \frac{1}{3}a^2}$

Hence the volume is $\frac{1}{3} \cdot \frac{\sqrt{3}}{4}a^2 \cdot \sqrt{b^2 - \frac{1}{3}a^2}$

$= \frac{a^2}{12}\sqrt{3b^2 - a^2}$

Now, consider a cross section of the centre. The red and blue lines are the same in both diagrams.



YB is the height of the pyramid $= \sqrt{b^2 - \frac{1}{3}a^2}$

Considering triangle ABY we have

$AB^2 = b^2 - \frac{1}{3}a^2 + \frac{1}{12}a^2$

$= b^2 - \frac{1}{4}a^2$

$\Rightarrow AB = \sqrt{b^2 - \frac{1}{4}a^2}$

Hence $\cos\theta = \dfrac{AY}{AB} = \dfrac{\frac{\sqrt{3}}{6}a}{\sqrt{b^2 - \frac{1}{4}a^2}}$

$$= \frac{a}{\sqrt{12b^2 - 3a^2}}$$

So $\theta = \arccos \dfrac{a}{\sqrt{12b^2 - 3a^2}}$

Then $XC^2 = AC^2 \cdot \sin^2\theta$

$$= \frac{3}{4}a^2 \cdot \left(1 - \frac{a^2}{12b^2 - 3a^2}\right)$$

$$= \frac{3}{4}a^2 \left(\frac{12b^2 - 4a^2}{12b^2 - 3a^2}\right)$$

$$= \frac{3}{4}a^2 \cdot \frac{4(3b^2 - a^2)}{3(4b^2 - a^2)}$$

$$= \frac{a^2(3b^2 - a^2)}{4b^2 - a^2}$$

STEP I 2002 Q7

$$I+J=\int_0^a dx$$
$$=a$$

$$I-J=\int_0^a \frac{\cos x-\sin x}{\sin x+\cos x}\,dx=\Big[\ln|\sin x+\cos x|\Big]_0^a$$
$$=\ln|\sin a+\cos a|-0$$

Hence $2I=a+\ln(\sin a+\cos a)$ (no modulus needed as $0\le a<3\pi/4$)

(i) Let $I=\int_0^{\pi/2}\frac{\cos x}{p\sin x+q\cos x}\,dx$, $J=\int_0^{\pi/2}\frac{\sin x}{p\sin x+q\cos x}\,dx$

then $qI+pJ=\pi/2$

$$pI-qJ=\int_0^{\pi/2}\frac{p\cos x-q\sin x}{p\sin x+q\cos x}\,dx=\Big[\ln|p\sin x+q\cos x|\Big]_0^{\pi/2}$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; \ln p\quad \ln q$$

Hence $q^2I+pqJ=q\pi/2$

$p^2I-pqJ=p(\ln p-\ln q)$

$$\Rightarrow I=\frac{1}{p^2+q^2}\left(q\pi/2+p\ln\frac{p}{q}\right)$$

(ii) $I=\int_0^{\pi/2}\frac{\cos x+4}{3\sin x+4\cos x+25}\,dx$ $J=\int_0^{\pi/2}\frac{\sin x+3}{3\sin x+4\cos x+25}\,dx$

$4I+3J=\pi/2$

$$3I-4J=\int_0^{\pi/2}\frac{3\cos x-4\sin x}{3\sin x+4\cos x+25}\,dx$$
$$=\Big[\ln|3\sin x+4\cos x+25|\Big]_0^{\pi/2}$$
$$=\ln\frac{28}{29}$$

So $16I+12J=2\pi$

$9I-12J=3\ln\frac{28}{29}$

$$\Rightarrow I=\frac{1}{25}\left(2\pi+3\ln\frac{28}{29}\right)$$

STEP I 2002 Q8

$C_1 = c$

$C_2 = C_1(1+\alpha) - R_1$

$C_3 = [C_1(1+\alpha) - R_1](1+\alpha) - R_2$

$C_4 = (1+\alpha)^3 C_1 - (1+\alpha)^2 R_1 - (1+\alpha) R_2 - R_3$

Hence $C_{n+1} = (1+\alpha)^n c - \sum_{i=1}^{n} (1+\alpha)^{n-i} R_i$

If the loan is paid off in N years with $R_k = (1+\alpha)^k r$, then

$$(1+\alpha)^N c - \sum_{i=1}^{n} (1+\alpha)^{n-i} (1+\alpha)^i r = 0$$

$$\Rightarrow (1+\alpha)^N c - N(1+\alpha)^N r = 0$$

$$\Rightarrow \quad r = \frac{c}{N}$$

Equal repayments over N years has

$$(1+\alpha)^N c - \sum_{i=1}^{N} (1+\alpha)^{n-i} R = 0$$

$$\Rightarrow c(1+\alpha)^N - R \cdot \frac{1-(1+\alpha)^N}{1-(1+\alpha)} = 0$$

$$\Rightarrow c\alpha(1+\alpha)^N + R[1-(1+\alpha)^N] = 0$$

$$\Rightarrow \frac{R}{c} = \frac{\alpha(1+\alpha)^N}{(1+\alpha)^N - 1}$$

$\alpha = \frac{1}{50}$, $N=4$, $\frac{R}{c} = \dfrac{\frac{1}{50}\cdot\left(1+\frac{1}{50}\right)^4}{\left(1+\frac{1}{50}\right)^4 - 1}$

Considering terms up to order $\frac{1}{50^2}$,

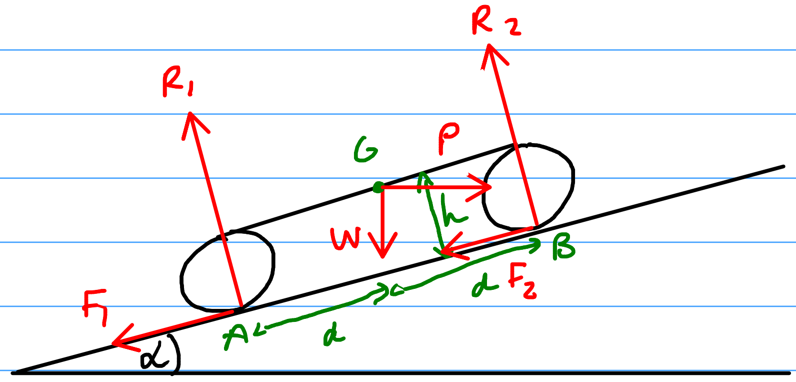
$$\frac{R}{C} \approx \frac{\frac{1}{50}\left(1+\frac{4}{50}\right)}{\left(1+\frac{4}{50}+\frac{6}{50^2}\right)-1}$$

$$= \frac{50+4}{200+6}$$

$$= \frac{54}{206}$$

$$= \frac{27}{103}$$

STEP I 2002 Q9



(i) $R_1 = R_2 = R$

$M(G)$: $dR + hF_2 = dR + hF_1$

$\Rightarrow F_1 = F_2$

(ii) $M(A)$: $2dR_2 + hW\sin\alpha = dW\cos\alpha + hP\cos\alpha + dP\sin\alpha$

$\Rightarrow 2dR_2 = W(d\cos\alpha - h\sin\alpha) + P(h\cos\alpha + d\sin\alpha)$

$R_2 \geq 0$, so $W(d\cos\alpha - h\sin\alpha) + P(h\cos\alpha + d\sin\alpha) \geq 0$

$\Rightarrow W \dfrac{h\sin\alpha - d\cos\alpha}{h\cos\alpha + d\sin\alpha} \leq P$ ($\cos\alpha, \sin\alpha \geq 0$ as $0 < \alpha < \pi/2$)

$\Rightarrow W \cdot \dfrac{\tan\alpha - d/h}{1 + \tan\alpha . d/h} \leq P$

$\Rightarrow W \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \leq P$

$\Rightarrow W\tan(\alpha - \beta) \leq P$

$M(B)$: $2dR_1 + hP\cos\alpha = dW\cos\alpha + hW\sin\alpha + dP\sin\alpha$

$\Rightarrow 2dR_1 = W(d\cos\alpha + h\sin\alpha) + P(d\sin\alpha - h\cos\alpha)$

$R_1 \geq 0$, so $W(d\cos\alpha + h\sin\alpha) + P(d\sin\alpha - h\cos\alpha) \geq 0$

$\Rightarrow P \dfrac{h\cos\alpha - d\sin\alpha}{d\cos\alpha + h\sin\alpha} \leq W$ ($\cos\alpha, \sin\alpha \geq 0$ as before)

$\Rightarrow P \cdot \dfrac{1 - \tan\alpha\tan\beta}{\tan\beta + \tan\alpha} \leq W$

$$\Rightarrow P \cdot \frac{1}{\tan(\alpha+\beta)} \leq W$$

$$\Rightarrow P \leq W\tan(\alpha+\beta)$$

(provided $\alpha+\beta<\pi$, which is true otherwise the lorry topples over)

So $W\tan(\alpha-\beta) \leq P \leq W\tan(\alpha+\beta)$

2002 STEP I Q10

Because the collisions are perfectly elastic,

$$V_{n+1} - u = V_n + u$$
$$\Rightarrow V_{n+1} = V_n + 2u$$

Hence $\quad V_n = V + 2nu$

We have $d_n - d_{n+1} = ut_n$ by definition of the quantities.

Also, $d_n + d_{n+1} = v_n t_n$
$$= (v + 2nu) \cdot \frac{d_n - d_{n+1}}{u}$$

$$\Rightarrow u(d_n + d_{n+1}) = (v + 2nu)(d_n - d_{n+1})$$
$$\Rightarrow d_{n+1}(v + (2n+1)u) = d_n(v + (2n-1)u)$$

$$\Rightarrow d_{n+1} = \frac{v + (2n-1)u}{v + (2n+1)u}$$

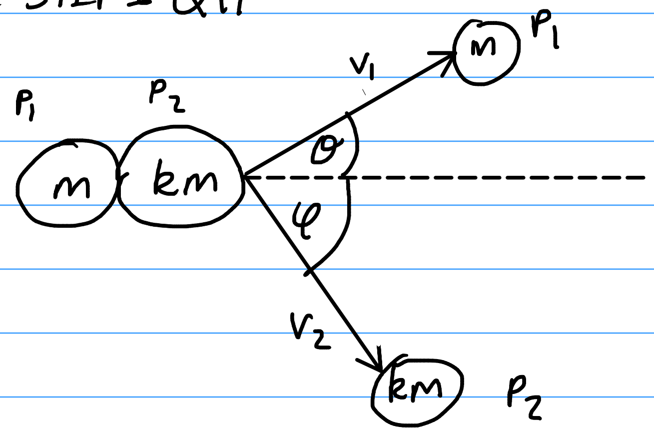
Hence $d_n = \dfrac{v + \cancel{(2n-3)u}}{v + (2n-1)u} \cdot \dfrac{v + \cancel{(2n-5)u}}{v + \cancel{(2n-3)u}} \cdot \;\cdots\; \cdot \dfrac{v+u}{v \cancel{+3u}} d_1$

$$= \frac{v+u}{v + (2n-1)u} d_1$$

If $v = u$, then $d_n = \frac{2v}{2nv} d_1$
$$= \frac{1}{n} d_1$$

Then $ut_n = d_n - d_{n+1}$
$$= d_1\left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= \frac{d_1}{n(n+1)}$$

2002 STEP I Q11



Conservation of energy gives $\frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}kmv_2^2$

$$\Rightarrow u^2 = v_1^2 + kv_2^2 \quad (1)$$

Conservation of momentum gives

$\leftrightarrow$ $$mu = mv_1\cos\theta + kmv_2\cos\phi$$

$$\Rightarrow u = v_1\cos\theta + kv_2\cos\phi \quad (2)$$

$\updownarrow$ $$0 = mv_1\sin\theta - kmv_2\sin\phi$$

$$\Rightarrow v_1\sin\theta = kv_2\sin\phi \quad (3)$$

Substituting (1) into (2) gives

$$v_1^2 + kv_2^2 = v_1^2\cos^2\theta + k^2v_2^2\cos^2\phi + 2kv_1v_2\cos\theta\cos\phi$$

$$\Rightarrow \left(\frac{v_1}{v_2}\right)^2 + k = \left(\frac{v_1}{v_2}\right)^2\cos^2\theta + k^2\cos^2\phi + 2k\frac{v_1}{v_2}\cos\theta\cos\phi$$

From (3), $\frac{v_1}{v_2} = \frac{k\sin\phi}{\sin\theta}$, so the above equation becomes

$$k^2\frac{\sin^2\phi}{\sin^2\theta} + k = k^2\frac{\sin^2\phi\cos^2\theta}{\sin^2\theta} + k^2\cos^2\phi + \frac{2k^2\sin\phi\cos\theta\cos\phi}{\sin\theta}$$

$$\Rightarrow k\sin^2\phi + \sin^2\theta = k\left[\sin^2\phi\cos^2\theta + 2\sin\theta\cos\theta\sin\phi\cos\phi + \sin^2\theta\cos^2\phi\right]$$

$$= k\left[(\sin\theta\cos\phi + \sin\phi\cos\theta)^2\right]$$

$$= k\sin^2(\theta+\phi), \text{ as required.}$$

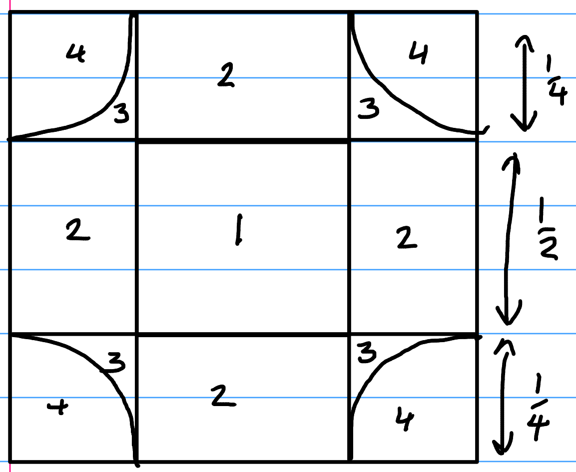
If $\theta + \phi = 90°$, then $\sin^2(\theta+\phi) = 1$, so

$$\sin^2\theta + k\sin^2\phi = k$$

$$\Rightarrow k = \frac{\sin^2\theta}{1-\sin^2\phi} = \frac{\sin^2\theta}{\cos^2\phi} = \frac{\sin^2\theta}{\sin^2\theta} \quad (\text{as } \theta = 90 - \phi)$$

$$= 1, \text{ as required.}$$

2002 STEP I Q12



$P(1) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$P(4) = \pi \times \left(\frac{1}{4}\right)^2 = \frac{\pi}{16}$

So those results confirm the suspicions Harry chooses at random.

Let X be average value of Harry's answers. Then

$$\begin{aligned} EX &= 1 \times \frac{1}{4} + 2 \times \frac{1}{2} + 3 \times \left(\frac{1}{4} - \frac{\pi}{16}\right) + 4 \times \frac{\pi}{16} \\ &= \frac{1}{4} + 1 + \frac{3}{4} - \frac{3\pi}{16} + \frac{4\pi}{16} \\ &= 2 + \frac{\pi}{16} \\ &\approx 2.2 \end{aligned}$$

If Harry was answering correctly, $EX = \frac{1}{4}(1+2+3+4)$
$= 2.5.$

$2 < 2.2 < 2.5$ so Harry is doing even worse than picking at random!

2002 STEP I Q13

$v \backslash u$	1	0	-1
1	1/9	1/6	2/9
-1	2/9	1/6	1/9

(i) $P(\text{both real}) = P(u^2 - 4v \geq 0)$

$= P(u^2 \geq 4v)$

$= P((u,v) \in \{(1,-1),(0,-1),(-1,-1)\})$

$= \frac{2}{9} + \frac{1}{6} + \frac{1}{9}$

$= \frac{1}{2}$

(ii) The larger root is $\dfrac{-u+\sqrt{u^2-4v}}{2}$

(u,v)	Root
$(1,-1)$	$\frac{-1+\sqrt{5}}{2}$
$(0,-1)$	1
$(-1,-1)$	$\frac{1+\sqrt{5}}{2}$

So $E(\text{root} \mid \text{real})$ is $\dfrac{\frac{2}{9}\left(\frac{-1+\sqrt{5}}{2}\right) + \frac{1}{6}\times 1 + \frac{1}{9}\left(\frac{1+\sqrt{5}}{2}\right)}{\frac{1}{2}}$

$= \frac{1}{18}\left(-4+4\sqrt{5}+6+2+2\sqrt{5}\right)$

$= \frac{1}{18}\left(4+6\sqrt{5}\right)$

$= \frac{1}{9}(2+3\sqrt{5})$

(iii) $x^3 + (u-2v)x^2 + (1-2uv)x + u = 0$

For all positive roots, $u < 0$ (negative y-intercept), so $u = -1$.

$v=-1 \Rightarrow x^3 + x^2 - x - 1 = 0$

$\Rightarrow (x-1)(x^2+2x+1) = 0$

$\Rightarrow (x-1)(x+1)^2 = 0$ $\#$

$v = 1 \Rightarrow x^3 - 3x^2 + 3x - 1 = 0$

$\Rightarrow (x-1)^3 = 0$ ✓ So probability is $\frac{2}{9}$

2002 STEP I Q14

$$P(\text{police arrive}) = \sum_{k=1}^{n} P(\text{correct on } k^{th} \text{ attempt}) P(\text{police arrive before } k^{th} \text{ attempt})$$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot 1 - e^{\frac{-\lambda(k-1)}{r}}$$

← because police arrive as soon as the first wrong attempt, and it then takes $\frac{(k-1)}{r}$ minutes until k^{th} attempt

$$= \frac{1}{n} \sum_{k=1}^{n} 1 - e^{\frac{-\lambda(k-1)}{r}}$$

$$= \frac{1}{n}\left(n - \sum_{k=1}^{n} e^{\frac{-\lambda(k-1)}{r}}\right)$$

$$= 1 - \frac{1}{n} e^{\frac{\lambda}{r}} \sum_{k=1}^{n} e^{-\lambda k/r}$$

$$= 1 - \frac{1}{n} \cancel{e^{\lambda/r}} \cdot \frac{\cancel{e^{-\lambda/r}}(1 - e^{-\lambda n/r})}{1 - e^{-\lambda r}}$$

$$= 1 - \frac{1 - e^{-\lambda n/r}}{n(1 - e^{-\lambda r})}$$

Now, P(successful on k^{th} attempt)

$$= \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$$

$$= \frac{(n-1)^{k-1}}{n^k}$$

$$\text{So } P(\text{not caught}) = \frac{1}{n} \sum_{k=1}^{\infty} \left(\frac{n-1}{n}\right)^{k-1} e^{-\lambda(k-1)/r}$$

$$= \frac{1}{n} \sum_{k=1}^{\infty} \left[\frac{n-1}{n} \cdot e^{-\lambda/r}\right]^{k-1}$$

$$= \frac{1}{n} \cdot \frac{1}{1 - \frac{n-1}{n} e^{-\lambda/r}}$$

$$= \frac{1}{n - (n-1)e^{-\lambda/r}}$$

$$\text{So } P(\text{caught}) = 1 - \frac{1}{n - (n-1)e^{-\lambda/r}}$$