STEP III 2001 Q1

$$y = \ln(x + \sqrt{x^2+1})$$

$$\frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

$$= \frac{\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}}{\sqrt{x^2+1} + x}$$

$$= \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d^2y}{dx^2} = -x(x^2+1)^{-3/2}$$

For $n=0$,

$$(x^2+1)(-x)(x^2+1)^{-3/2} + (2\times 0+1)x(x^2+1)^{-1/2} + 0^2 \ln(x+\sqrt{x^2+1})$$

$$= -x(x^2+1)^{-1/2} + x(x^2+1)^{-1/2}$$

$$= 0$$

Assume true for $n=k$, so

$$(x^2+1)y^{(k+2)} + (2k+1)xy^{(k+1)} + k^2 y^{(k)} = 0$$

Differentiating,

$$(x^2+1)y^{(k+3)} + 2xy^{(k+2)} + (2k+1)y^{(k+1)} + (2k+1)xy^{(k+2)} + k^2 y^{(k+1)} = 0$$

$$\Rightarrow (x^2+1)y^{(k+3)} + (2x + 2kx + x)y^{(k+2)} + (k^2+2k+1)y^{(k+1)} = 0$$

$$\Rightarrow (x^2+1)y^{(k+3)} + (2(k+1)+1)y^{(k+2)} + (k+1)^2 y^{(k+1)} = 0$$

So true for $n=k+1$. Hence true for $n \geq 0$ by induction.

For $x=0$, $y^{(n+2)} = -n^2 y^{(n)}$

$y(0)=0$, so all even derivatives are 0.

$y'(0)=1$, $\Rightarrow y^{(3)}(0) = -1 \Rightarrow y^{(5)}(0) = 9 \Rightarrow y^{(7)}(0) = 3^2 \times 5^2$

Hence the Maclaurin series is

$$x - \frac{x^3}{3!} + \frac{9x^5}{5!} - \frac{3^2 \times 5^2 x^7}{7!}$$

$$= x - \frac{x^3}{6} + \frac{3x^5}{2\times4\times5} - \frac{5x^7}{2\times4\times2\times7}$$

$$= x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112}$$

STEP II 2001 Q2

$$\cosh x = \frac{e^x+e^{-x}}{2}$$

$$y = \frac{e^x+e^{-x}}{2}$$

$$\Rightarrow 2y = e^x + e^{-x}$$

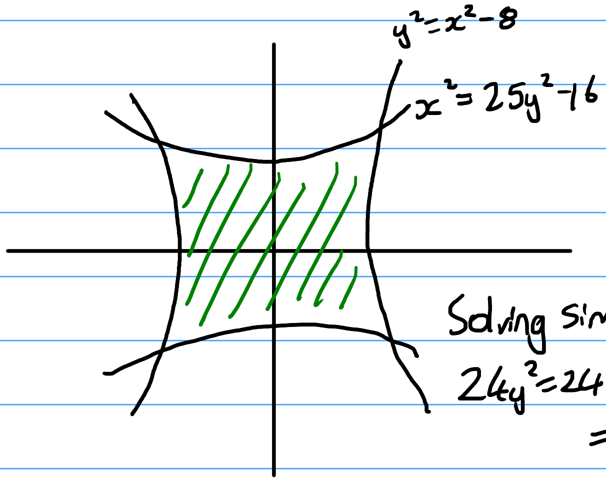
$$\Rightarrow e^{2x} - 2ye^x + 1 = 0$$

$$\Rightarrow e^x = \frac{2y \pm \sqrt{4y^2-4}}{2}$$ but $e^x \geqslant 0$, so use $^+$ve square root.

$$\Rightarrow e^x = y + \sqrt{y^2-1}$$

$$\Rightarrow x = \ln(y+\sqrt{y^2-1})$$

So $\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$

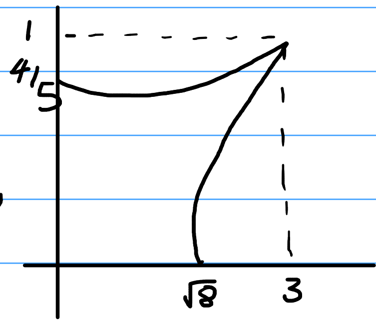


Solving simultaneously,

$24y^2 = 24 \Rightarrow y = \pm 1$

$\Rightarrow x = \pm 3$

Consider one quarter,



Hence one quarter of the total area is given by

$$\int_0^3 \frac{1}{5}\sqrt{x^2+16}\,dx - \int_{\sqrt{8}}^{3} \sqrt{x^2-8}\,dx$$

Using $x = 4\sinh u$ $\qquad$ $x = \sqrt{8}\cosh v$

$dx = 4\cosh u\,du$ $\qquad$ $dx = \sqrt{8}\sinh v\,dv$

$$= \frac{1}{5}\int_{\text{arsinh}\,0}^{\text{arsinh}\,3/4} 4\sqrt{1+\sinh^2 u}\;4\cosh u\,du - \int_{\text{arcosh}\,1}^{\text{arcosh}\,3/\sqrt{8}} \sqrt{8}\sqrt{\cosh^2 v - 1}\cdot\sqrt{8}\sinh v\,dv$$

$$= \frac{16}{5}\int_0^{\operatorname{arsinh}3/4} \cosh^2 u \, du - 8\int_0^{\operatorname{arcosh}3/\sqrt{8}} \sinh^2 v \, dv$$

Using $\cosh 2u = 2\cosh^2 u - 1$

$$= 1 + 2\sinh^2 u$$

$$= \frac{16}{5}\int_0^{\operatorname{arsinh}3/4} \frac{1}{2} + \frac{1}{2}\cosh 2u \, du - 8\int_0^{\operatorname{arcosh}3/\sqrt{8}} -\frac{1}{2} + \frac{1}{2}\cosh 2v \, dv$$

$$= \frac{16}{5}\left[\frac{u}{2} + \frac{1}{4}\sinh 2u\right]_0^{\operatorname{arsinh}3/4} - 8\left[-\frac{1}{2}v + \frac{1}{4}\sinh 2v\right]_0^{\operatorname{arcosh}3/\sqrt{8}}$$

$$= \frac{16}{5}\left[\frac{u}{2} + \frac{1}{2}\sinh u\cosh u\right]_0^{\operatorname{arsinh}3/4} - 8\left[-\frac{1}{2}v + \frac{1}{2}\sinh v\cosh v\right]_0^{\operatorname{arcosh}3/\sqrt{8}}$$

$$= \frac{16}{5}\left[\left(\frac{1}{2}\operatorname{arsinh}\frac{3}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\right) - (0+0)\right] - 8\left[\left(-\frac{1}{2}\operatorname{arcosh}\frac{3}{\sqrt{8}} + \frac{1}{2}\cdot\frac{1}{\sqrt{8}}\cdot\frac{3}{\sqrt{8}}\right) - (0+0)\right]$$

$$= \frac{8}{5}\operatorname{arsinh}\frac{3}{4} + \frac{3}{2} + 4\operatorname{arcosh}\frac{3}{\sqrt{8}} - \frac{3}{2}$$

$$= \frac{8}{5}\ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right) + 4\ln\left(\frac{3}{\sqrt{8}} + \sqrt{\frac{9}{8} - 1}\right)$$

$$= \frac{8}{5}\ln 2 + 4\ln\left(\frac{4}{\sqrt{8}}\right)$$

$$= \frac{8}{5}\ln 2 + 2\ln 2$$

$$= \frac{18}{5}\ln 2$$

So total area is $4 \times \frac{18}{5}\ln 2$

$$= \frac{72}{5}\ln 2, \text{ as required.}$$

STEP III 2001 Q3

$x^2 - bx + c = 0$

$\Rightarrow x = \dfrac{b \pm \sqrt{b^2-4c}}{2}$

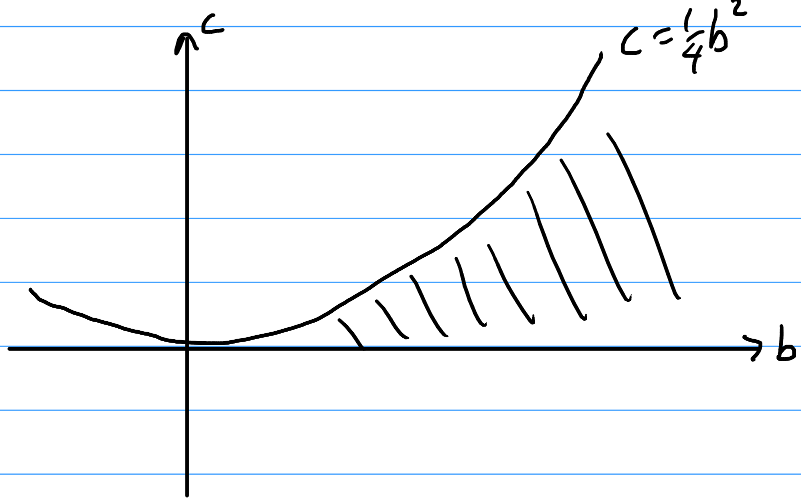
If $b>0$ and $b^2 \geqslant 4c > 0$, then $0 \leq \sqrt{b^2-4c} < b$

$\Rightarrow$ smaller root is $b - \sqrt{b^2-4c} > 0$,

so both roots are real and positive.

If both roots are real and positive then $b^2-4c \geqslant 0 \Rightarrow b^2 \geqslant 4c$. If $b \leq 0$ then $b - \sqrt{b^2-4c} \leq 0$ #. So $b>0$. Finally if $4c=0$ then smaller root is $\dfrac{b-\sqrt{b^2}}{2} = 0$ #. So $4c>0$.

$b>0$ and $b^2 \geqslant 4c > 0$ gives



Now assume the roots are real and less than 1 in magnitude.

Then $b + \sqrt{b^2-4c} < 2$

$\sqrt{b^2-4c} < 2-b$

So we get $2-b>0$

$\Rightarrow b<2$

$b - \sqrt{b^2-4c} > -2$

$\sqrt{b^2-4c} < b+2$

$b+2>0$

$\Rightarrow b>-2$

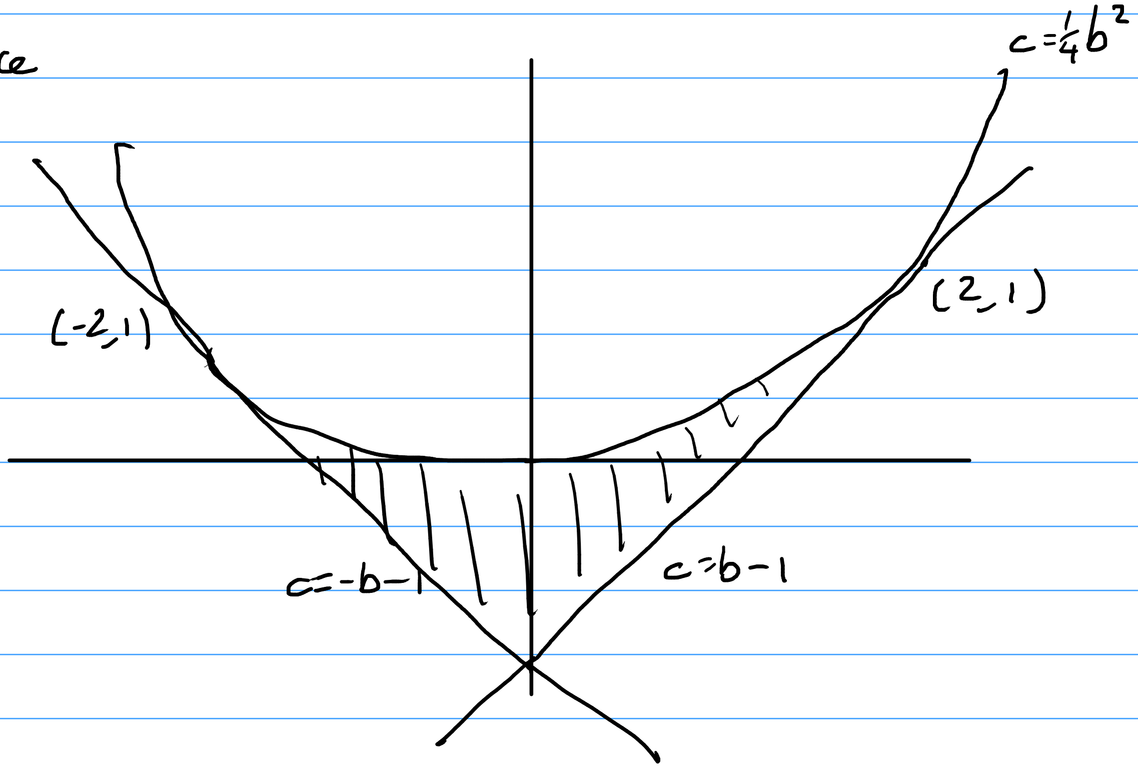
As both sides are positive, squaring we obtain

$$b^2-4c<4-4b+b^2 \qquad b^2-4c<b^2+4b+4$$

$$\Rightarrow \quad -c<1-b \qquad -c<b+1$$

$$\Rightarrow \quad c>b-1 \qquad c>-1-b$$

Hence



Noting that $b-1=\frac{1}{4}b^2 \Rightarrow b^2-4b+4=0$

$\Rightarrow (b-2)^2=0$

$\Rightarrow b=2$ only, so tangent (and symmetry for $c=-b-1$)

STEP III 2001 Q4

(i) $g(x) = \frac{2x}{1+x^2}$

$$g'(x) = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2}$$

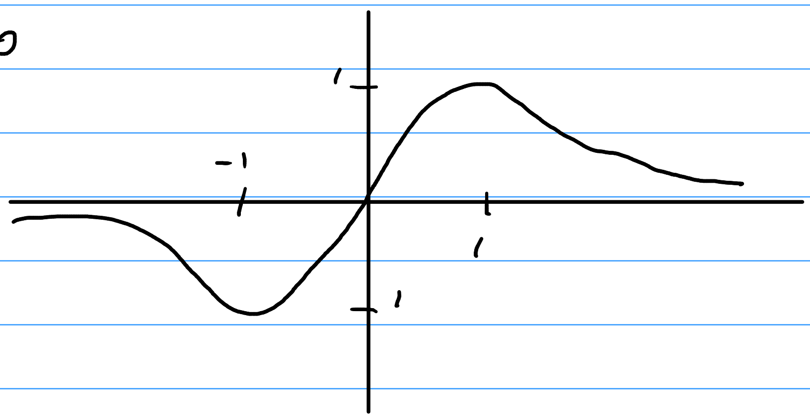
$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

$$g'(x) \begin{cases} <0 & |x|>1 \\ =0 & |x|=1 \\ >0 & |x|<1 \end{cases}$$

$g(1) = 1$, $g(-1) = -1$

$\lim_{x\to\infty} g(x) = 0$

$\lim_{x\to-\infty} g(x) = 0$



The range of g is $-1 \le g(x) \le 1$

(ii) $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $\theta = 2\tan^{-1}(x)$, so $x = \tan\frac{\theta}{2}$

$\Rightarrow \sin\theta = \frac{2x}{1+x^2}$ (t formula)

Hence $f(x) = \sin^{-1}(\sin\theta)$

For $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin^{-1}(\sin\theta) = \theta$

$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Rightarrow 2\tan^{-1}x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Rightarrow \quad x \in [-1, 1]$

So, for $x \in [-1, 1]$, $f(x) = \theta$

$= 2\tan^{-1}x.$

For $x \geq 1$, $2\tan^{-1}(x) \in [\pi/2, \pi)$

$\Rightarrow \quad \theta \in [\pi/2, \pi)$

$\Rightarrow \pi - \theta \in (0, \pi/2]$

so, as $\sin\theta = \sin(\pi - \theta)$ we have

$f(x) = \sin^{-1}(\sin\theta)$

$= \pi - \theta$

$= \pi - 2\tan^{-1}(x)$

For $x \leq -1$, $2\tan^{-1}(x) \in (-\pi, -\frac{\pi}{2}]$

$\Rightarrow \quad \theta \in (-\pi, -\frac{\pi}{2}]$

$\Rightarrow -\theta - \pi \in (-\frac{\pi}{2}, 0]$

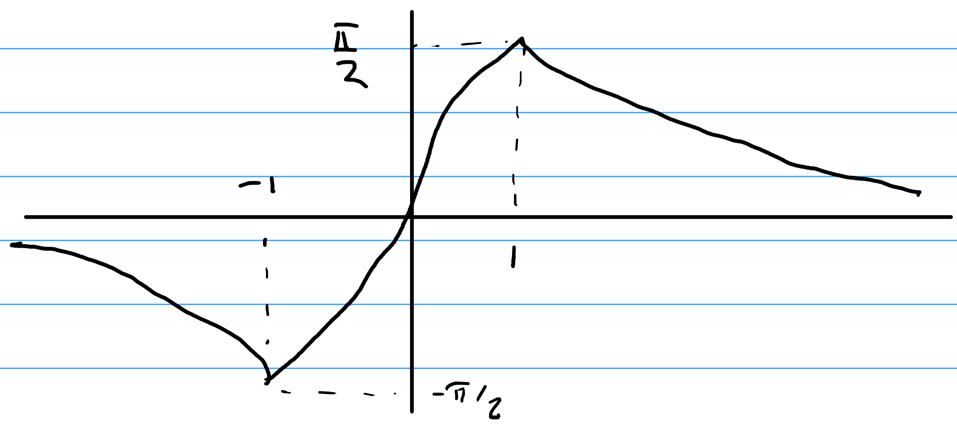
and as $\sin\theta = \sin(-\theta - \pi)$,

$f(x) = \sin^{-1}(\sin\theta)$

$= -\pi - \theta$

$= -\pi - 2\tan^{-1}(x)$

So,



STEP III 2001 Q5

$y = x^3 + px + q$

$y' = 3x^2 + p.$

If $p \geq 0$ then $y' \geq 0$ $\forall x$, and so y is strictly increasing (apart from possibly at $x=0$ if $p=0$). Hence y crosses the x axis at most once. Because $\lim_{x \to -\infty} y = -\infty$ and $\lim_{x \to \infty} y = +\infty$, the graph crosses the x axis at least once. Hence the graph crosses the axis exactly once and so $x^3 + px + q = 0$ has exactly one solution if $p \geq 0$.

$x = at^2$, $y = 2at$

$$\frac{dy}{dx} = 2a \cdot \frac{1}{2at}$$
$$= \frac{1}{t}$$

Hence gradient of normal is $-t$.

So equation of normal is $(y - 2at) = -t(x - at^2)$ for each t.

Passes through (h,k), so

$$k - 2at = -t(h - at^2)$$
$$\Rightarrow at^3 + (2a-h)t - k = 0$$
$$\Rightarrow t^3 + \frac{2a-h}{a}t - \frac{k}{a} = 0$$

This is the same form as previously, so this has exactly one real solution if $\frac{2a-h}{a} \geq 0 \Leftrightarrow h \leq 2a$ $(a \neq 0)$.

For exactly two normals, the cubic must have a turning point on the x-axis. The turning point satisfies $3at^2 + (2a-h) = 0$

$$\Rightarrow t = \pm\sqrt{\frac{h-2a}{3a}}$$

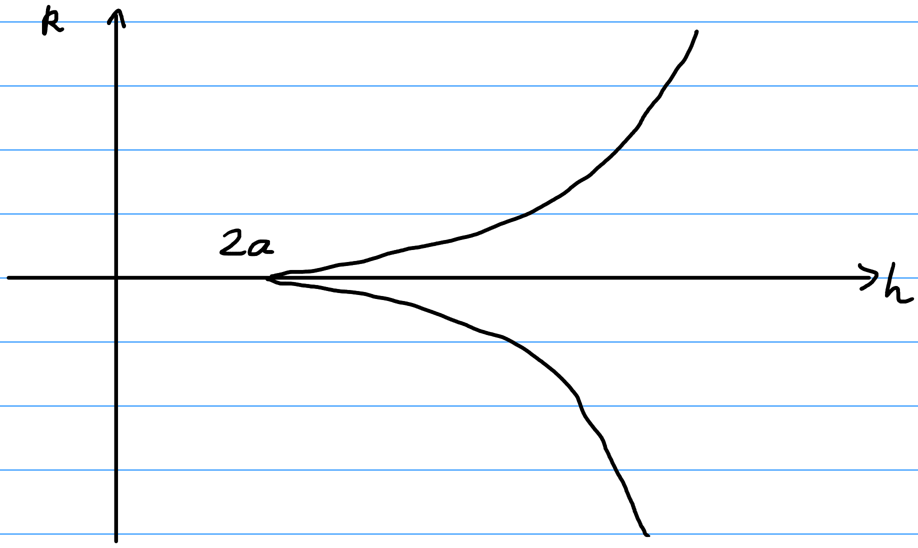
This is also a solution of the cubic, so

$$\pm a\left(\frac{h-2a}{3a}\right)^{3/2} \pm (2a-h)\left(\frac{h-2a}{3a}\right)^{1/2} - k = 0$$

$$\Rightarrow \pm \frac{1}{3}\frac{(h-2a)^{3/2}}{(3a)^{1/2}} \mp \frac{(h-2a)^{3/2}}{(3a)^{1/2}} - k = 0$$

$$\Rightarrow k = \mp \frac{2}{3}\frac{(h-2a)^{3/2}}{(3a)^{1/2}}$$

$$\Rightarrow k = \pm \frac{2}{3}\sqrt{\frac{(h-2a)^3}{3a}}$$



STEP III 2001 Q6

$A=(a,0,0)$, $B=(0,b,0)$, $C=(0,0,c)$.

A vector equation of the plane is $\underline{r}\cdot\begin{pmatrix}1/a\\1/b\\1/c\end{pmatrix}=1$, so $\begin{pmatrix}1/a\\1/b\\1/c\end{pmatrix}$ is normal to the plane.

OM is $\lambda\begin{pmatrix}a\\b\\c\end{pmatrix}$. $\quad \lambda\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1/a\\1/b\\1/c\end{pmatrix}=1$

$\Rightarrow 3\lambda=1$

$\Rightarrow \lambda=1/3$

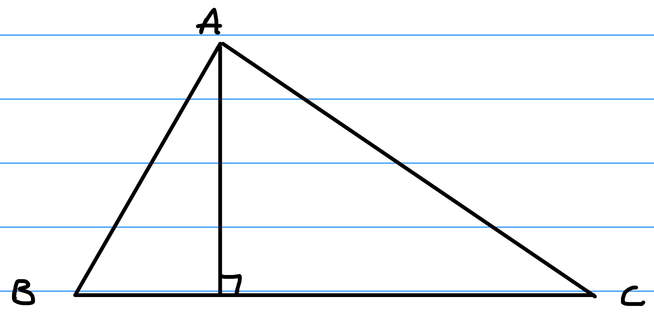
$\Rightarrow$ OM meets the plane at $\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right)$.

The perpendicular to the plane from O is $\lambda\begin{pmatrix}1/a\\1/b\\1/c\end{pmatrix}$. Using the equation of the plane, $\lambda\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=1$

$$\Rightarrow \lambda=\frac{a^2b^2c^2}{a^2b^2+a^2c^2+b^2c^2}$$

So the perpendicular to the plane from O meets the plane at P where

$$\overrightarrow{OP}=\frac{a^2b^2c^2}{a^2b^2+a^2c^2+b^2c^2}\begin{pmatrix}1/a\\1/b\\1/c\end{pmatrix}$$



$\overrightarrow{BC}=\begin{pmatrix}0\\-b\\c\end{pmatrix}$

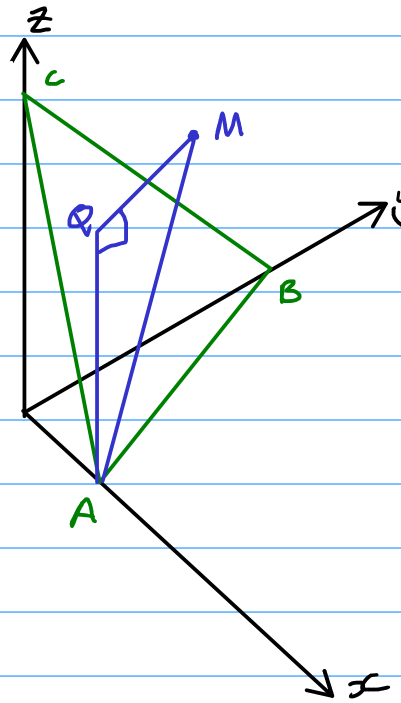
$$\overrightarrow{AP}=\frac{a^2b^2c^2}{a^2b^2+a^2c^2+b^2c^2}\begin{pmatrix}1/a\\1/b\\1/c\end{pmatrix}-\begin{pmatrix}a\\0\\0\end{pmatrix}$$

Hence $\overrightarrow{BC}\cdot\overrightarrow{AP}=0-\dfrac{a^2b^2c^2}{a^2b^2+a^2c^2+b^2c^2}+\dfrac{a^2b^2c^2}{a^2b^2+a^2c^2+b^2c^2}$

$=0.$

Hence AP is perpendicular to BC. Similarly BP is perpendicular to AC, and CP is perpendicular to AB.

Hence P must lie on all of the altitudes of the triangle, so is the orthocentre.



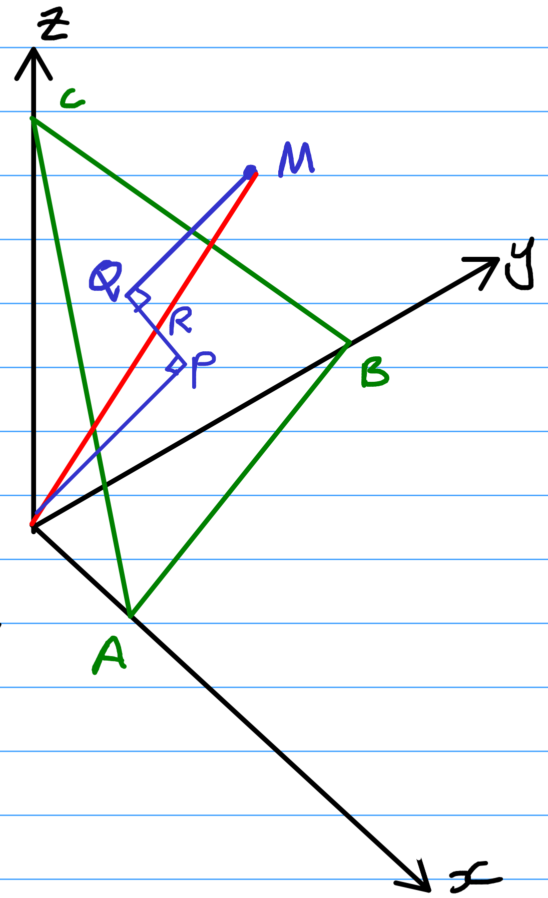
Let the perpendicular to the plane from M intersect the plane at Q. Then we have

$$MQ^2 + QA^2 = MA^2$$
$$= \left| \begin{pmatrix} \frac{1}{2}a \\ -\frac{1}{2}b \\ -\frac{1}{2}c \end{pmatrix} \right|^2$$
$$= \frac{1}{4}(a^2+b^2+c^2)$$

Hence $QA^2 = \frac{1}{4}(a^2+b^2+c^2) - MQ^2$.

The RHS is symmetric in a, b, and c, so we get $QA^2 = QB^2 = QC^2$, so Q is the circumcentre.

Finally, MQ and OP both meet the plane at right angles and so are parallel. Hence triangles OPR and MRQ are similar, where R is where OM intersects the plane. Because $\overrightarrow{OR} = \frac{2}{3}\overrightarrow{OM}$ $\left(\frac{1}{3}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \frac{1}{2}\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right)$, we have $OR = 2 \times RM$ so $PR = 2 \times QR$. So the centroid R lies on the line connecting P and Q and divides this segment in the ratio 2:1, as required.



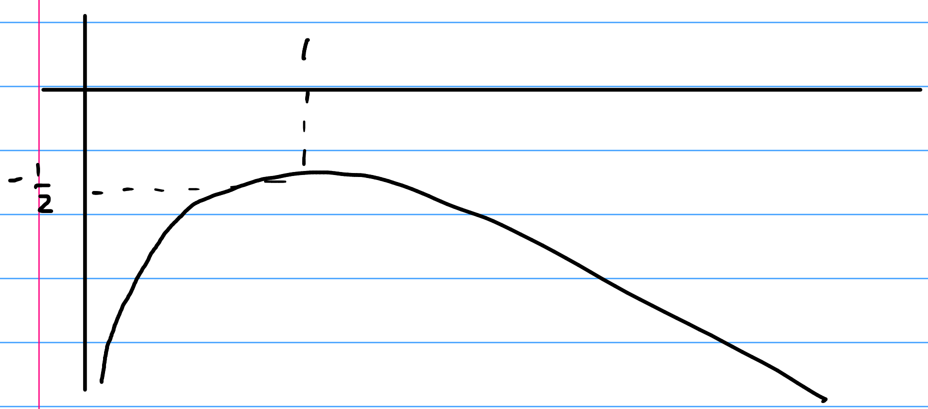
STEP III 2001 Q7

$y = \ln x - \frac{1}{2}x^2$

$y' = \frac{1}{x} - x = 0 \Rightarrow x^2 - 1 = 0$

$\Rightarrow x = 1,\ y = -\frac{1}{2}$ (as $x > 0$)

$\lim_{x \to 0^+} y = -\infty, \quad \lim_{x \to \infty} y = -\infty$



$\frac{dy}{dx} = \frac{2xy}{x^2-1}$

$\Rightarrow \frac{1}{y}\,dy = \frac{2x}{x^2-1}\,dx$

$\Rightarrow \ln y = \ln|x^2-1| + \ln k$

$\Rightarrow y = A(x^2-1)$

$y = 0 \Rightarrow x = \pm 1$ ✓

vertex is at $(0, -A)$ ✓

The curve which is orthogonal to these parabolas passes through the point (p, q). At this point, for the parabola, $y' = \frac{2pq}{p^2-1}$, so the gradiant of the normal is $\frac{1-p^2}{2pq}$.

So, for the normal curve, we have $\frac{dy}{dx} = \frac{1-x^2}{2xy}$

$\Rightarrow y\,dy = \left(\frac{1}{2x} - \frac{x}{2}\right)dx$

$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}\ln|x| - \frac{1}{4}x^2 + c$

$\Rightarrow y^2 = \ln|x| - \frac{1}{2}x^2 + c$

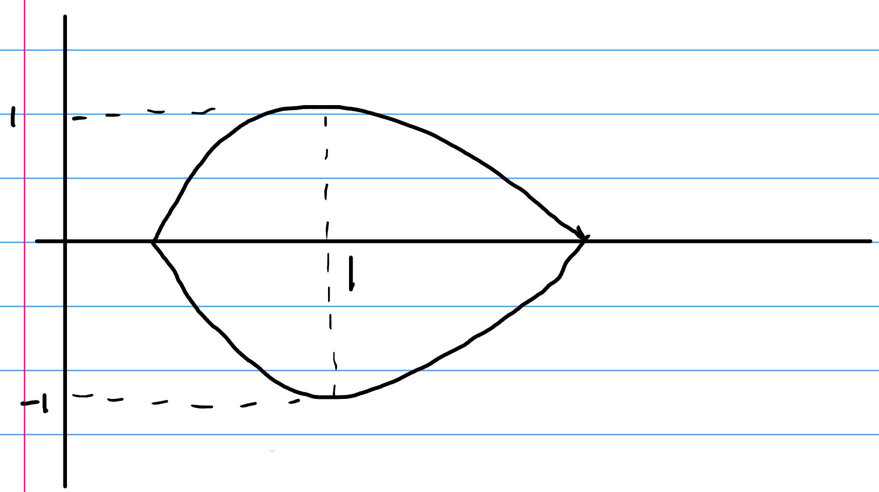
Passes through $(1,1) \Rightarrow 1 = 0 - \frac{1}{2} + c$

$\Rightarrow c = \frac{3}{2}$

So $y^2 = \ln|x| - \frac{1}{2}x^2 + \frac{3}{2}$

We previously sketched $y = \ln x - \frac{1}{2}x^2$.

$y = \pm\sqrt{\ln x - \frac{1}{2}x^2 + \frac{3}{2}}$ has domain only where $\ln x - \frac{1}{2}x^2 > -\frac{3}{2}$. The x coordinate of the turning point is unchanged as $\sqrt{}$ is increasing.



STEP III 2001 Q8

(i) Let $z = x + iy$ with $x, y \in \mathbb{R}$

$|z-(1+i)|^2 = 2$

$\Rightarrow |(x-1)+i(y-1)|^2 = 2$

$\Rightarrow (x-1)^2 + (y-1)^2 = 2$

$|z-(1-i)|^2 = 2|z-1|^2$

$\Rightarrow (x-1)^2 + (y+1)^2 = 2(x-1)^2 + 2y^2$

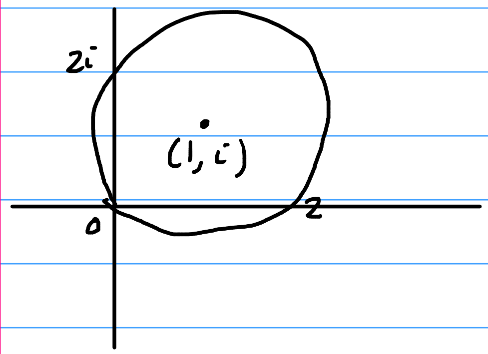
$\Rightarrow y^2 + 2y + 1 = (x-1)^2 + 2y^2$

$\Rightarrow 1 = (x-1)^2 + y^2 - 2y$

$\Rightarrow 1 = (x-1)^2 + (y-1)^2 - 1$

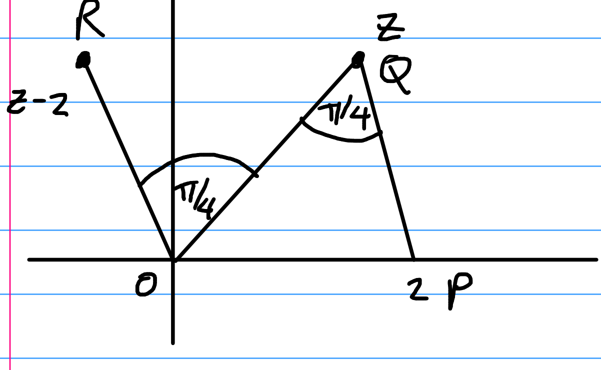
$\Rightarrow 2 = (x-1)^2 + (y-1)^2$

So the describe the same locus.



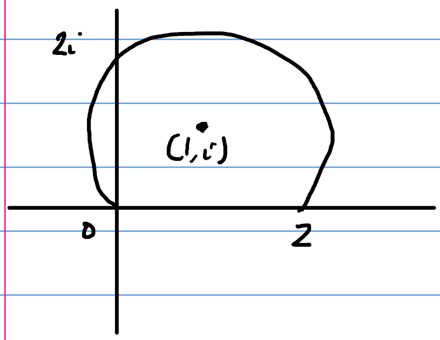
(ii) $\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{4}$

$\Rightarrow \arg(z-2) - \arg(z) = \pi/4$



$\angle QOR = \pi/4$ by definition.
Then $\angle PZO = \pi/4$ by alternate angles.
Hence z sweeps out the arc of a circle (as angles in the same segment are equal).
OP is a chord of the circle in (i), and $2i$ lies on this circle and is clearly in the new locus, so it is the same circle.

However, if $\mathrm{Im}(z) < 0$, then $\arg(z-2) - \arg(z) < 0$, so clearly not $\pi/4$. So it is only the part of the circle above the real axis.



(iii) $z = \frac{2}{w}$, so $\left|\frac{2}{w} - (1+i)\right|^2 = 2$

$\Rightarrow \left|\frac{2-w-iw}{w}\right|^2 = 2$

$\Rightarrow |2-w-wi|^2 = 2|w|^2$

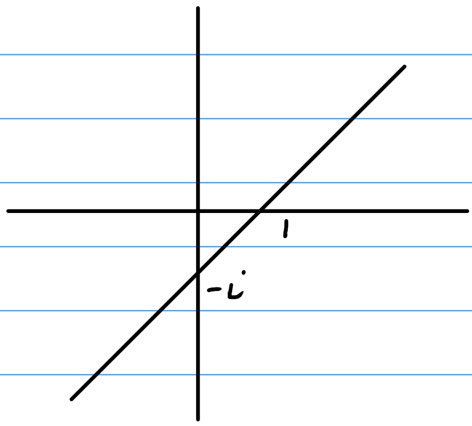
$\Rightarrow |w(1+i)-2| = \sqrt{2}|w|$

$\Rightarrow |1+i|\left|w-\frac{2}{1+i}\right| = \sqrt{2}|w|$

$\Rightarrow \sqrt{2}\left|w-\frac{2(1-i)}{2}\right| = \sqrt{2}|w|$

$\Rightarrow |w-(1-i)| = |w|$

which is the perpendicular bisector of $(1-i)$ and the origin; the line $y=x-1$.

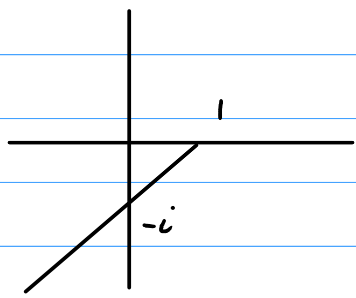


$\mathrm{Arg}\left(\frac{z-2}{z}\right) = \arg\left(1-\frac{2}{z}\right)$

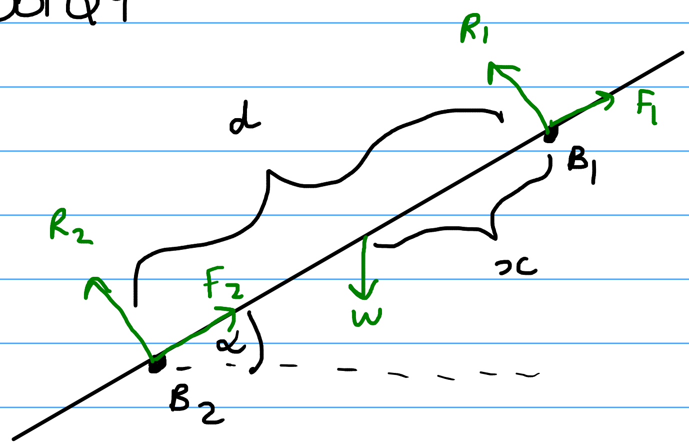
$= \arg(1-w) = \pi/4$

$\Rightarrow \arg(w-1) = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$

which is the half-line



STEP III 2001 Q9



When the beam has moved a distance x,

$M(B_2)$: $(d-x)W\cos\alpha = dR_1$

$$\Rightarrow R_1 = W\frac{d-x}{d}\cos\alpha$$

$M(B_1)$: $xW\cos\alpha = dR_2$

$$\Rightarrow R_2 = \frac{x}{d}W\cos\alpha$$

Hence, in the $\swarrow$ direction, we have

$$\frac{W}{g}\ddot{x} = W\sin\alpha - W\mu_1\frac{d-x}{d}\cos\alpha - W\mu_2\frac{x}{d}\cos\alpha$$

$$\Rightarrow \ddot{x} = g\left(\sin\alpha + \frac{x}{d}\cos\alpha(\mu_1-\mu_2) - \mu_1\cos\alpha\right)$$

$$\Rightarrow \ddot{x} + \frac{g\cos\alpha}{d}(\mu_2-\mu_1)x = g(\sin\alpha - \mu_1\cos\alpha)$$

The complementary function is $x = A\sin(\omega t + c)$ with $\omega^2 = \frac{g\cos\alpha}{d}(\mu_2-\mu_1)$

The particular integral is $x = \dfrac{g(\sin\alpha - \mu_1\cos\alpha)}{\frac{g\cos\alpha}{d}(\mu_2-\mu_1)}$

$$= \frac{\sin\alpha - \mu_1\cos\alpha}{\cos\alpha(\mu_2-\mu_1)}d$$

$$= \frac{\sin\alpha - \mu_1\cos\alpha}{\cos\alpha(2\tan\alpha - \mu_1 - \mu_1)}d$$

$$= \frac{\sin\alpha - \mu_1\cos\alpha}{2(\sin\alpha - \mu_1\cos\alpha)}d$$

$$= \frac{d}{2}$$

So $x = A\sin(\omega t + c) + \frac{1}{2}d$

$x(0) = 0 \Rightarrow c = -\arcsin\left(\frac{d}{2A}\right)$

$\dot{x} = A\omega\cos(\omega t + c)$

$\dot{x}(0) = 0 \Rightarrow c = -\frac{\pi}{2}$

$\Rightarrow -\arcsin\left(\frac{d}{2A}\right) = -\frac{\pi}{2}$

$\Rightarrow \frac{d}{2A} = 1$

$\Rightarrow A = \frac{1}{2}d$

So $x = \frac{1}{2}d\left(\sin(\omega t - \frac{\pi}{2}) + 1\right)$

Setting $x = d$ gives $2 = \sin(\omega t - \pi/2) + 1$

$\Rightarrow \sin(\omega t - \pi/2) = 1$

$\Rightarrow \omega t - \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow t = \frac{\pi}{\omega}$

At this point, $\dot{x} = A\omega\cos(\omega t - \pi/2)$

$= A\omega\cos(\pi/2) = 0$, as required.

So the time taken is π/ω

$= \pi\sqrt{\frac{d}{g\cos\alpha(\mu_2 - \mu_1)}}$, as required.

STEP III 2001 Q10

$A_{\text{rel}}C = v_1 - u, \quad B_{\text{rel}}C = v_2 - u$

So $(v_1 - u)\cdot(v_2 - u) = 0$ and $|v_1 - u|^2 = |v_2 - u|^2$

The first equation becomes

$$v_1\cdot v_2 - v_1\cdot u - v_2\cdot u + u\cdot u = 0 \qquad (1)$$

The second equation becomes

$$v_1\cdot v_1 - 2v_1\cdot u + \cancel{u\cdot u} = v_2\cdot v_2 - 2v_2\cdot u + \cancel{u\cdot u} \qquad (2)$$

Now, $|u - \tfrac{1}{2}(v_1+v_2)|^2$

$$= u\cdot u - u\cdot v_1 - u\cdot v_2 + \tfrac{1}{4}(v_1\cdot v_1 + 2v_1\cdot v_2 + v_2\cdot v_2)$$

$$= -v_1\cdot v_2 + \tfrac{1}{4}(v_1\cdot v_1 + 2v_1\cdot v_2 + v_2\cdot v_2) \qquad \text{by (1)}$$

$$= \tfrac{1}{4}(v_1\cdot v_1 - 2v_1\cdot v_2 + v_2\cdot v_2)$$

$$= |\tfrac{1}{2}(v_1 - v_2)|^2, \text{ as required.} \qquad (*)$$

Then

$$(u - \tfrac{1}{2}(v_1+v_2))\cdot(v_1 - v_2)$$

$$= u\cdot v_1 - u\cdot v_2 - \tfrac{1}{2}v_1\cdot v_1 + \tfrac{1}{2}v_2\cdot v_2$$

$$= \tfrac{1}{2}\left[2u\cdot v_1 - v_1\cdot v_1 - (2u\cdot v_2 - v_2\cdot v_2)\right]$$

$$= 0 \qquad \text{by (2)} \qquad (\dagger)$$

Hence, $u - \tfrac{1}{2}(v_1+v_2)$ is of fixed non-zero magnitude (by $(*)$ and $v_1 \neq v_2$), and is perpendicular to $v_1 - v_2$ (by $(\dagger)$), so there exist exactly two values for $u - \tfrac{1}{2}(v_1+v_2)$, so exactly two values for u.

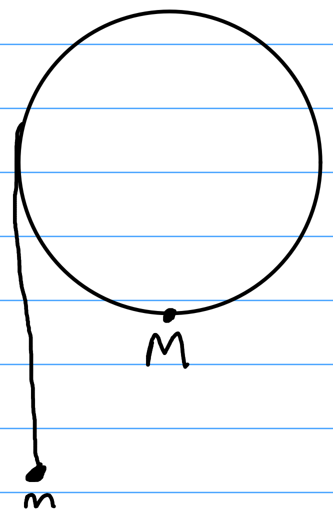
If $|v_1| = |v_2|$ and $v_1\cdot v_2 = 0$, then we have

$$u\cdot u - u\cdot(v_1+v_2) = 0 \quad (1)$$

$$u\cdot(v_1 - v_2) = 0 \quad (2)$$

$u = 0$ and $u = v_1 + v_2$ both work, and we know there are exactly two solutions, so $u = v_1 + v_2$ is the only non-zero solution.

STEP III 2001 Q11



If the cylinder rotates through θ, the decrease in GPE is $mga\theta - Mga(1-\cos\theta)$

The increase in KE is $\frac{1}{2}I\omega^2 + \frac{1}{2}ma^2\omega^2 + \frac{1}{2}Ma^2\omega^2$

So, $\frac{1}{2}\omega^2(I + ma^2 + Ma^2) = ga(m\theta - M + M\cos\theta)$

$$\Rightarrow \omega = \sqrt{\frac{2ga(m\theta - M + M\cos\theta)}{I + a^2(m+M)}}$$

The cylinder never stops if $m\theta - M + M\cos\theta > 0$

$$\Rightarrow \frac{m}{M} > \frac{1-\cos\theta}{\theta} \quad \text{for all } \theta.$$

Differentiating wrt θ, we obtain $\dfrac{\theta\sin\theta + \cos\theta - 1}{\theta^2} = 0$

$$\Rightarrow \theta\sin\theta + \cos\theta = 1$$

Setting $t = \tan(\frac{1}{2}\theta)$, $\theta \cdot \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$

$$\Rightarrow 2t\theta = 2$$

$$\Rightarrow \theta = t$$

$$\Rightarrow \theta = \tan(\tfrac{1}{2}\theta)$$

Let α be the solution of this equation, so $\alpha = \tan(\frac{1}{2}\alpha)$, and α minimises $\frac{1-\cos\theta}{\theta}$

Then $\dfrac{1-\cos\alpha}{\alpha}$

$$= \frac{1 - \frac{1-t^2}{1+t^2}}{t}$$

$$= \frac{2t}{1+t^2}$$

$$= \sin\alpha$$

And so we require $\frac{m}{M} > \sin\alpha$,

STEP III 2001 Q12

(i) Each arrangement of the Bs and Ws is equally likely. Once the Ws are placed in a line, there are $w+1$ locations to put the Bs, of which w occur before the final white. So the number of blacks befor the final white $\sim B(b, \frac{w}{w+1})$ with expected value $\frac{bw}{w+1}$.

(ii) The probability that the number of times a black ball is removed before the first white ball is removed is k is $\left(\frac{b}{w+b}\right)^k \cdot \frac{w}{w+b}$.

Hence the expected number is $\sum_{k=0}^{\infty} \frac{w}{w+b}\left(\frac{b}{w+b}\right)^k \cdot k$

$$= \frac{w}{w+b} \sum_{k=0}^{\infty} k p^k \qquad \text{where } p = \frac{b}{w+b}$$

$$= \frac{w}{w+b} \sum_{k=0}^{\infty} p \times \frac{d}{dp}(p^k)$$

$$= \frac{w}{w+b} \cdot p \cdot \frac{d}{dp} \sum_{k=0}^{\infty} p^k$$

$$= \frac{w}{w+p} \cdot p \cdot \frac{d}{dp} \frac{1}{1-p}$$

$$= \frac{w}{w+b} \cdot p \cdot \frac{1}{(1-p)^2}$$

$$= \frac{w}{w+b} \cdot \frac{b}{w+b} \cdot \frac{1}{\left[\frac{w}{w+b}\right]^2}$$

$$= \frac{b}{w}, \text{ as required.}$$

Hence the total expected number of times a black ball is removed is

$$\sum_{r=1}^{w} \frac{b}{r}$$

STEP III 2001 Q13

(i) $p_{TT}=1$ because if the next coin is heads, A wins, and if it's tails, we are still in the same situation.

If the first two coins are HT, with probability $\frac{1}{2}$, the next is heads, in which case the last two coins are TH, so we are in the same situation as if the first two coins were TH. Similarly there is probability $\frac{1}{2}$ of being in the TT situation. So $p_{HT} = \frac{1}{2}p_{TH} + \frac{1}{2}p_{TT}$

$$= \frac{1}{2}p_{TH} + \frac{1}{2} \quad (1)$$

We also have $p_{TH} = 0 + \frac{1}{2}p_{HT}$ (2)

$$p_{HH} = \frac{1}{2}p_{HT} + \frac{1}{2}p_{HH} \quad (3)$$

(3) gives $p_{HH} = p_{HT}$, then (2) gives $p_{TH} = \frac{1}{2}p_{HT}$, then (1) gives $p_{HT} = \frac{1}{4}p_{HT} + \frac{1}{2}$

$\Rightarrow p_{HT} = \frac{2}{3}$

$\Rightarrow p_{HH} = \frac{2}{3}$

$\Rightarrow p_{TH} = \frac{1}{3}$

So $P(A \text{ wins}) = \frac{1}{4}(1 + \frac{1}{3} + \frac{2}{3} + \frac{2}{3})$

$= \frac{2}{3}$

(ii) Now B&C. If the first two coins are HH, then C wins (as above). Otherwise, there is at least one tail. Hence, the first run of HH must be preceded by T, so THH occurs before HHT can. So $P(C \text{ wins}) = P(HH \text{ first}) = 1/4$, $P(B \text{ wins}) = 3/4$

(iii) Let the p be P(C wins).

$p_{HH} = 1$ (1)

$p_{HT} = \frac{1}{2}p_{TH} + 0$ (2)

$p_{TH} = \frac{1}{2}p_{HH} + \frac{1}{2}p_{HT}$ (3)

$p_{TT} = \frac{1}{2}p_{TH} + \frac{1}{2}p_{TT}$

(1) & (2) into (3)

$\Rightarrow p_{TH} = \frac{1}{2} + \frac{1}{4}p_{TH}$

$\Rightarrow p_{TH} = \frac{2}{3}$

$\Rightarrow p_{HT} = \frac{1}{3}$

$\Rightarrow p_{TT} = \frac{1}{2}$

So $P(C \text{ wins}) = \frac{1}{4}(1 + \frac{2}{3} + \frac{1}{3} + \frac{1}{2})$

$= \frac{5}{8}$.

D vs A is similar to B vs C. If the first two coins are TT, A wins. Otherwise, D wins. So $P(D \text{ wins}) = 3/4$.

STEP III 2001 Q14

The PDF of X is $\begin{cases} 1/a & 0<x<a \\ 0 & o/w \end{cases}$

Clearly $EX = \frac{1}{2}$

$$EX^2 = \int_0^a \frac{1}{a}x^2\,dx = \left[\frac{x^3}{3a}\right]_0^a$$

$$= a^2/3$$

So $\operatorname{Var}X = \frac{a^2}{3} - \left(\frac{a}{2}\right)^2$

$$= \frac{a^2}{12}$$

$$V = \frac{X_1^2 + X_2^2}{2} - \left(\frac{X_1 + X_2}{2}\right)^2$$

$$= \frac{1}{4}(2X_1^2 + 2X_2^2 - X_1^2 - X_2^2 - 2X_1X_2)$$

$$= \frac{1}{4}(X_1 - X_2)^2$$

$$E2V = \frac{1}{2}E(X_1 - X_2)^2$$

$$= \frac{1}{2}E(X_1^2 + X_2^2 - 2EX_1EX_2) \qquad \text{(by independence)}$$

$$= \frac{1}{2}\left(\frac{a^2}{3} + \frac{a^2}{3} - 2\frac{a}{2}\cdot\frac{a}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{6}a^2\right)$$

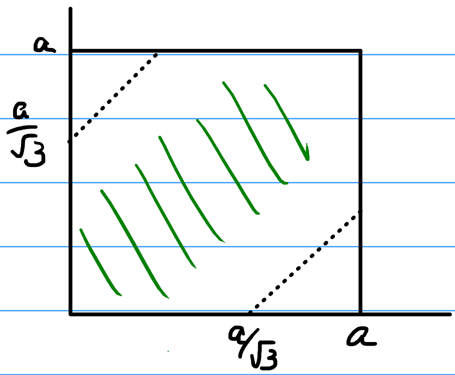
$= \frac{1}{12}a^2$, so $2V$ is unbiased for the variance of X.

$$P\left(V < \frac{1}{12}a^2\right)$$

$$= P\left(\frac{1}{4}(X_1 - X_2)^2 < \frac{1}{12}a^2\right)$$

$$= P\left(|X_1 - X_2| < \frac{1}{\sqrt{3}}a\right)$$

$$= \frac{1}{a^2}\cdot\left[a^2 - 2\cdot\frac{1}{2}\cdot\left(1 - \frac{1}{\sqrt{3}}\right)^2 a^2\right]$$



$$= 1 - \left(1 - \frac{\sqrt{3}}{3}\right)^2$$

$$= 1 - \left(1 - \frac{2\sqrt{3}}{3} + \frac{1}{3}\right)$$

$$= \frac{2\sqrt{3} - 1}{3}$$

$\sqrt{3} \approx 1.7$, so this is $\approx \frac{3.4 - 1}{3}$

$$= 0.8$$