STEP II 2001 Q1

1) $(1-x)^{1/2} \approx 1 - \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-x)^2$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

(i) $\sqrt{\frac{99}{100}} = \frac{3}{10}\sqrt{11} \approx 1 - \frac{1}{200} - \frac{1}{8} \times \frac{1}{10{,}000}$

$$\Rightarrow 3\sqrt{11} \approx 10 - \frac{1}{20} - \frac{1}{8000}$$

$$= \frac{80{,}000 - 400 - 1}{8000}$$

$$= 79599/8000$$

$$\Rightarrow \sqrt{11} \approx 79599/24000$$

Next term is $\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}(-x)^3 = -\frac{1}{16}x^3$

Error is approximately $\frac{10}{3} \times (\frac{-1}{16}) \times 10^{-6}$

$$= -\frac{5}{24} \times 10^{-6}$$

$$\approx -2 \times 10^{-7}$$

(ii) Note $1111 \times 9 = 9999$, so setting $x = \frac{1}{10{,}000}$ gives

$$\sqrt{\frac{9999}{10000}} = \frac{3}{100}\sqrt{1111}$$

The expansion gives $1 - \frac{1}{2} \times \frac{1}{10{,}000} - \frac{1}{8} \times \frac{1}{10^8}$

$$\Rightarrow 3\sqrt{1111} \approx 100 - \frac{1}{2} \times \frac{1}{100} - \frac{1}{8} \times \frac{1}{10^6}$$

$$= \frac{1}{8 \times 10^6}(800{,}000{,}000 - 40{,}000 - 1)$$

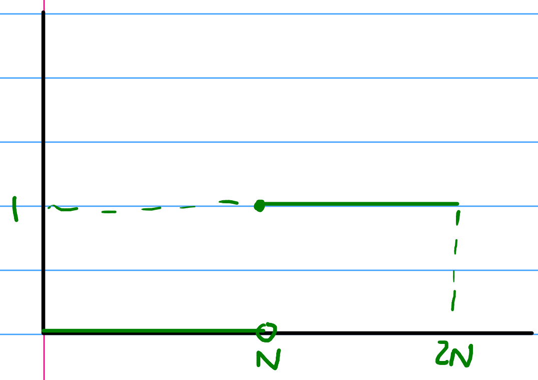
$$\Rightarrow \sqrt{1111} \approx \frac{799{,}959{,}999}{24{,}000{,}000}$$

The error is $\approx \frac{1}{16} \times (10^{-4})^3 \times \frac{100}{3}$

$$\approx \frac{25}{12} \times 10^{-12}$$

$$\approx 2 \times 10^{-12}$$

STEP II 2001 Q2



(i) $\sum_{k=1}^{2N} (-1)^{[k/N]} k = (1+2+3+ \text{—} + N-1) - (N + N+1 + \text{———} + 2N-1) + 2N$

$= \frac{N(N-1)}{2} - \frac{N}{2}(3N-1) + 2N$

$= N\left(\frac{1}{2}N - \frac{1}{2} - \frac{3N}{2} + \frac{1}{2} + 2\right)$

$= 2N - N^2$

(ii) $\sum_{k=1}^{2N} (-1)^{[k/N]} = \left(\frac{1}{2} + \frac{1}{4} + \text{———} + \frac{1}{2^{N-1}}\right) - \left(\frac{1}{2^N} + \text{———} + \frac{1}{2^{2N-1}}\right) + \frac{1}{2^{2N}}$

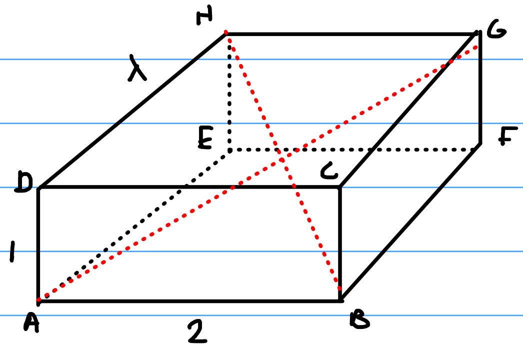
$= \frac{\frac{1}{2}\left(1 - \frac{1}{2^{N-1}}\right)}{\frac{1}{2}} - \frac{\frac{1}{2^N}\left(1 - \frac{1}{2^N}\right)}{\frac{1}{2}} + \frac{1}{2^{2N}}$

$= 1 - \frac{1}{2^{N-1}} - \frac{1}{2^{N-1}} + \frac{1}{2^{2N-1}} + \frac{1}{2^{2N}}$

$= 1 - 2^{-N+2} + 2^{-2N+1} + 2^{-2N}$

So $\lim_{N \to \infty} S_N = 1$.

STEP II 2001 Q3



Let x be the point of intersection of the diagonals. Then

$$Ax = \sqrt{\left(\tfrac{1}{2}\right)^2 + 1^2 + \left(\tfrac{\lambda}{2}\right)^2}$$
$$= \tfrac{1}{2}\sqrt{5+\lambda^2}$$

Then $\cos H x G = \dfrac{2 \times \frac{1}{4}(5+\lambda^2) - 4}{2 \times \frac{1}{4}(5+\lambda^2)}$

$$= \frac{5+\lambda^2 - 8}{5+\lambda^2}$$

$$= \frac{3-\lambda^2}{5+\lambda^2}$$

If this is positive, then HxG is acute. Else GxB is acute and $GxB = \pi - HxG \Rightarrow$
$\cos GxB = -\cos HxG = \dfrac{\lambda^2 - 3}{5+\lambda^2}$

So $\cos\alpha = \left|\dfrac{3-\lambda^2}{5+\lambda^2}\right|$

Volume $= 2\lambda$

$SA = 2+2+2\lambda+2\lambda+\lambda+\lambda$
$\quad = 4+6\lambda$

So the ratio is $\dfrac{2\lambda}{4+6\lambda} = \dfrac{\lambda}{2+3\lambda}$

Note that $R'(\lambda) = \dfrac{2+3\lambda-3\lambda}{(2+3\lambda)^2} > 0$ so R is increasing.

$R(0) = 0$, and $\lim_{\lambda\to\infty} R(\lambda) = \frac{1}{3}$, so $0 < R < \frac{1}{3}$ for all $\lambda > 0$.

If $R = 1/4$, then $\dfrac{\lambda}{2+3\lambda} = 1/4 \Rightarrow \lambda = 2$

So $R \geq 4 \Rightarrow \lambda \geq 2$.

Now $\lambda \geqslant 2 \Rightarrow \cos\alpha = \frac{\lambda^2 - 3}{5 + \lambda^2}$

Differentiating wrt λ^2, we obtain $\frac{(5+\lambda^2) - (\lambda^2 - 3)}{(5+\lambda^2)^2}$

$$= \frac{2}{(5+\lambda^2)} > 0$$

So $\cos\alpha$ is increasing in λ^2 and so increasing in λ. Hence α is decreasing in λ, so takes a maximum at $\lambda = 2$, then

$$\cos\alpha = \frac{4-1}{5+4}$$

$$= 1/9$$

$$\Rightarrow \alpha \leq \arccos(1/9)$$

STEP II 2001 Q4

$$f(x) = P\sin x + Q\sin 2x + R\sin 3x$$
$$= P\sin x + 2Q\sin x\cos x + R\sin x(4\cos^2 x - 1)$$
$$= \sin x(P + 2Q\cos x + 4R\cos^2 x - R)$$
$$= \sin x(4R\cos^2 x + 2Q\cos x + (P-R))$$

Clearly $x = m\pi$, $m \in \mathbb{Z}$ is a solution of $f(x)=0$.
Now $4R\cos^2 x + 2Q\cos x + (P-R) = 0$ has solutions for $\cos x$ only if

$$(2Q)^2 - 4(4R)(P-R) \geqslant 0$$
$$\Rightarrow \quad Q^2 \geqslant 4R(P-R)$$

So if $Q^2 < 4R(P-R)$ then $4R\cos^2 x + 2Q\cos x + (P-R) = 0$ has no solutions, so the only solutions of $f(x)=0$ are $x = m\pi$.

Now let $g(x) = \sin 2nx + \sin 4nx + \sin 6nx$

$$= \sin 2nx + 2\sin 2nx\cos 2nx - \sin 2nx(4\cos^2 2nx - 1)$$
$$= \sin 2nx(1 + 2\cos 2nx - 4\cos^2 2nx + 1)$$
$$= -2\sin 2nx(2\cos^2 2nx - \cos 2nx - 1)$$
$$= -2\sin 2nx(2\cos 2nx + 1)(\cos 2nx - 1)$$

$\sin 2nx = 0 \Rightarrow x = \frac{k\pi}{2n}$ for $k = 1, 2, 3, \_\_, (n-1)$

So the largest is $\frac{(n-1)\pi}{2n} = \frac{\pi}{2} - \frac{\pi}{2n}$

$\cos 2nx = 1 \Rightarrow \sin 2nx = 0$ so the solutions are a subset of before.

$\cos 2nx = -\frac{1}{2} \Rightarrow 2nx = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}, \ldots \quad 2k\pi \pm \frac{2\pi}{3}$

$\Rightarrow \quad x = \frac{\pi}{6n}, \_\_ \quad \_\_, \quad \frac{k\pi}{n} \pm \frac{\pi}{3n}$

If n is even, setting $k = n/2$ gets

$$x = \frac{\pi}{2} - \frac{\pi}{3n} \qquad \left(\frac{\pi}{2} + \frac{\pi}{3n} \text{ not in range}\right).$$

Setting $k = \frac{n}{2} + 1$ gives $\frac{\pi}{2} + \frac{\pi}{n} - \frac{\pi}{3n} = \frac{\pi}{2} + \frac{2\pi}{3n}$ not in range.

Since $\frac{\pi}{2} - \frac{\pi}{3n} > \frac{\pi}{2} - \frac{\pi}{2n}$, for n even the largest root is $\frac{\pi}{2} - \frac{\pi}{3n}$

If n is odd, setting $k = \left(\frac{n-1}{2}\right)$ gets

$$\frac{(n-1)\pi}{2n} + \frac{\pi}{3n}$$

$$= \frac{\pi}{2} - \frac{\pi}{6n} > \frac{\pi}{2} - \frac{\pi}{3n}$$

Setting $k = \left(\frac{n+1}{2}\right)$ gets

$$\frac{(n+1)\pi}{2n} - \frac{\pi}{3n}$$

$$= \frac{\pi}{2} + \frac{\pi}{6n} \text{ not in range.}$$

So for n odd, the largest root is $\frac{\pi}{2} - \frac{\pi}{6n}$

STEP II 2001 Q5

$$\frac{dy}{dx} = x(1-x^2)e^{-x^2} \qquad = (x-x^3)e^{-x^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = 0, 1, -1$$

$$\frac{d^2y}{dx^2} = (1-3x^2)e^{-x^2} - 2x(x-x^3)e^{-x^2}$$
$$= e^{-x^2}(2x^4 - 5x^2 + 1)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1,-1} = e^{-1}(2-5+1) < 0 \text{ so maximum point}$$

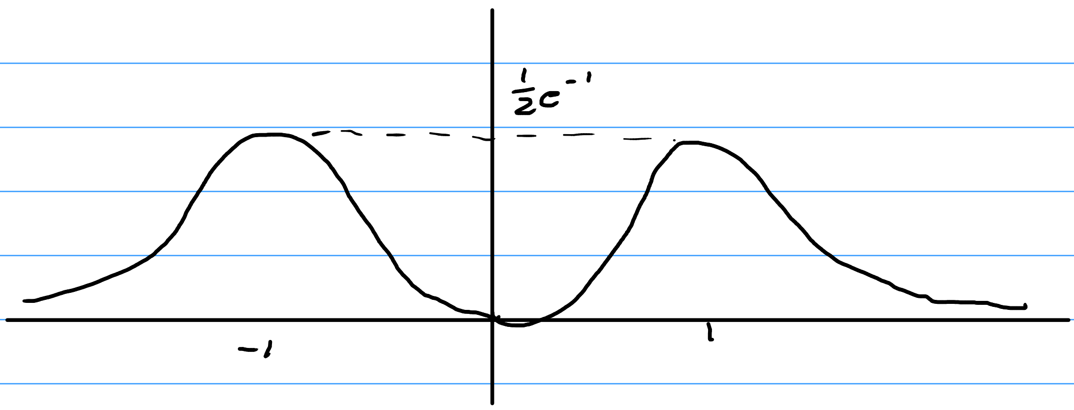
$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 1(1) > 0 \text{ so minimum point.}$$

$$y = \int xe^{-x^2} - x^3e^{-x^2}\,dx$$

$u \quad -x^2 \qquad v' \quad xe^{-x^2}$
$u' \quad -2x \qquad v \quad -\frac{1}{2}e^{-x^2}$

$$= -\frac{1}{2}e^{-x^2} + \frac{1}{2}xe^{-x^2} - \int xe^{-x^2}\,dx$$
$$= e^{-x^2}\left(-\frac{1}{2} + \frac{1}{2}x + \frac{1}{2}\right) + c$$
$$= \frac{1}{2}x^2e^{-x^2} + c$$

$y(0) = 0 \Rightarrow c = 0$
$y(1) = \frac{1}{2}e^{-1}$
$y(-1) = \frac{1}{2}e^{-1}$
So other stationary point is $(-1, \frac{1}{2}e^{-1})$



$$\frac{dy}{dx} = x(1-x^2)e^{-x^3}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = 0, 1, -1$$

$$\frac{d^2y}{dx^2} = e^{-x^3}(1-3x^2-3x^2(x-x^3))$$
$$= e^{-x^3}(1-3x^2-3x^3+3x^5)$$

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 1$, so minimum point (at $(0,0)$, as we are told C_2 passes through the origin)

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = e^{-1}(1-3-3+3) < 0$ so maximum point.

Note that $\frac{x(1-x^2)e^{-x^2}}{x(1-x^2)e^{-x^3}} = e^{x^3-x^2}$

$= e^{x^2(x-1)} < 1$ for $0<x<1$.

So the gradient of C_2 is larger than that of C_1 for $0<x<1$. Both pass through $(0,0)$ so C_2 has a larger y value at $x=1$. So $k > \frac{1}{2}e^{-1}$.

STEP II 2001 Q6

$$\int_0^1 \frac{x^4}{1+x^2}\,dx = \int_0^1 \frac{x^4+2x^2+1}{1+x^2} - \frac{1+x^2}{1+x^2} - \frac{x^2}{1+x^2}\,dx$$

$$= \int_0^1 \frac{(x^2+1)^2}{1+x^2} - 1 - \frac{x^2+1}{1+x^2} + \frac{1}{1+x^2}\,dx$$

$$= \int_0^1 1+x^2 - 1 - 1 + \frac{1}{1+x^2}\,dx$$

$$= \left[\tfrac{1}{3}x^3 - x + \arctan x\right]_0^1$$

$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

$$\int_0^1 x^3 \arctan\left(\frac{1-x}{1+x}\right) dx$$

u $\arctan\left(\frac{1-x}{1+x}\right)$ v' x^3

u' $\frac{d}{dx}\arctan\left(\frac{1-x}{1+x}\right)$ v $\frac{1}{4}x^4$

Note $\frac{d}{dx}\arctan\frac{1-x}{1+x} = \frac{-(1+x)-(1-x)}{(1+x)^2}\cdot\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}$

$$= \frac{-2}{(1+x)^2}\cdot\frac{(1+x)^2}{(1+x)^2+(1-x)^2}$$

$$= \frac{-2}{2x^2+2}$$

$$= \frac{-1}{1+x^2}$$

So $\int_0^1 x^3\arctan\left(\frac{1-x}{1+x}\right)dx = \left[\frac{1}{4}x^4\arctan\left(\frac{1-x}{1+x}\right)\right]_0^1 + \frac{1}{4}\int_0^1 \frac{x^4}{1+x^2}\,dx$

$$= (0-0) + \frac{1}{4}\left(\frac{\pi}{4}-\frac{2}{3}\right)$$

$$= \frac{\pi}{16} - \frac{1}{6}$$

$$\int_0^1 \frac{(1-y)^3}{(1+y)^5} \arctan y$$

Set $y = \frac{1-x}{1+x}$, $\frac{dy}{dx} = \frac{-2}{(1+x)^2}$

$$\Rightarrow dy = \frac{-2}{(1+x)^2} dx$$

Note $1-y = \frac{1+x-1+x}{1+x}$

$$= \frac{2x}{1+x}$$

$$1+y = \frac{1+x+1-x}{1+x}$$

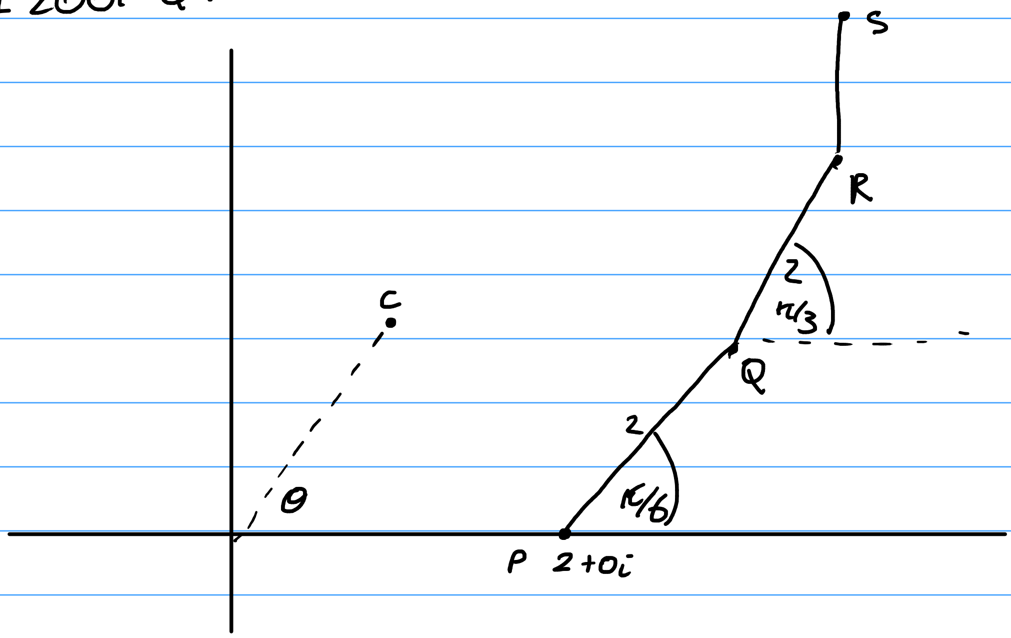
$$= \frac{2}{1+x}$$

$$= \int_1^0 \left(\frac{2x}{1+x}\right)^3 \cdot \left(\frac{1+x}{2}\right)^5 \cdot \arctan\left(\frac{1-x}{1+x}\right) \cdot \frac{-2}{(1+x)^2} dx$$

$$= \frac{1}{2}\int_0^1 x^3 \arctan\left(\frac{1-x}{1+x}\right) dx$$

$$= \frac{\pi}{32} - \frac{1}{12}$$

STEP II 2001 Q7



$Q = 2 + 2(\cos\frac{\pi}{6} + i\sin\pi/6)$

$= 2 + 2(\frac{\sqrt{3}}{2} + \frac{i}{2})$

$= 2 + \sqrt{3} + i$

$R = 2+\sqrt{3}+i + 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$= 2+\sqrt{3}+i+2(\frac{1}{2}+i\frac{\sqrt{3}}{2})$

$= 3+\sqrt{3}+i(1+\sqrt{3})$

$S = 3+\sqrt{3}+i(1+\sqrt{3}) + 2i$

$= (3+\sqrt{3}) + i(3+\sqrt{3})$

$= (3+\sqrt{3})(1+i)$

C lies on the perpendicular bisector of OP so lies on $x=1$, so $C = 1+\lambda i$.

$|OC|^2 = 1+\lambda^2$

$|QC|^2 = (1+\sqrt{3})^2 + (\lambda-1)^2$

$= 1+2\sqrt{3}+3+\lambda^2-2\lambda+1$

$= \lambda^2-2\lambda+5+2\sqrt{3}$

$|OC|^2 = |QC|^2 \Rightarrow 1+\lambda^2 = \lambda^2-2\lambda+5+2\sqrt{3}$

$\Rightarrow \lambda = 2+2\sqrt{3}$

So C is $1+(2+2\sqrt{3})i$

$\arg c = \arctan(2+\sqrt{3})$

$\Rightarrow \tan\theta = 2+\sqrt{3}$

Hence $\dfrac{\sin^2\theta}{1-\sin^2\theta} = (2+\sqrt{3})^2$

$\Rightarrow \sin^2\theta = (7+4\sqrt{3})(1-\sin^2\theta)$

$\Rightarrow \sin^2\theta = \dfrac{7+4\sqrt{3}}{8+4\sqrt{3}}$

$= \dfrac{(7+4\sqrt{3})(8-4\sqrt{3})}{64-48}$

$= \dfrac{(7+4\sqrt{3})(2-\sqrt{3})}{4}$

$= \dfrac{2+\sqrt{3}}{4}$

$\cos^2\theta = 1-\sin^2\theta = \dfrac{2-\sqrt{3}}{4}$

So $s' = (3+\sqrt{3})(1+i)\cdot(\cos(\pi-\theta) + i\sin(\pi-\theta))$

$= (3+\sqrt{3})(1+i)\cdot(-\cos\theta + i\sin\theta)$

$= \dfrac{(3+\sqrt{3})(1+i)}{2}\left(-\sqrt{2-\sqrt{3}} + i\sqrt{2+\sqrt{3}}\right)$

$= \dfrac{(3\sqrt{2}+\sqrt{6})}{2\sqrt{2}}\left(-\sqrt{2-\sqrt{3}} + i\sqrt{2+\sqrt{3}} - i\sqrt{2-\sqrt{3}} - \sqrt{2+\sqrt{3}}\right)$

$= \dfrac{(3\sqrt{2}+\sqrt{6})}{2\sqrt{2}}\left(-\left(\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}\right) + i\left(\sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}}\right)\right)$

Note that $\left(\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}\right)\right)^2 = \frac{1}{2}\left(2+\sqrt{3}+2-\sqrt{3}+2(4-3)\right)$

$= 3$, and $\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}} > 0$, so

$\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}\right) = \sqrt{3}$

Similarly $\left(\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}}\right)\right)^2 = \frac{1}{2}\left(2+\sqrt{3}+2-\sqrt{3}-2(4-3)\right)$

$= 1$, and $\sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}} > 0$, so

$\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}}\right) = 1$

So $s' = \dfrac{(3\sqrt{2}+\sqrt{6})}{2}(-\sqrt{3}+i)$

$= -\frac{1}{2}(3\sqrt{2}+\sqrt{6})(\sqrt{3}-i)$, as required.

STEP II 2001 Q8

(i) $f(x) = 1$

(ii) $F(n) = \int_0^n f(x)\,dx$

$= n\int_0^1 f(x)\,dx$ by the property

$= n(F(1) - F(0))$

$= nF(1)$

(iii) $\frac{dy}{dx} + f(x)y = 0$

$\frac{-1}{y}dy = f(x)\,dx$

$-\ln y = \int^x f(u)\,du$

$y = e^{-\int^x f(u)\,du}$

So $y(n) = e^{-\int^n f(u)\,du}$

$= e^{-nF(1)}$

$= (e^{-F(1)})^n \longrightarrow 0$ as $n \longrightarrow \infty$ because $F(1) > 0$ as $F(0) = 0$ and $F'(x) = f(x) > 0$.

STEP II 2008 Q9

$$W = \int_0^x \alpha e^{-\beta y} dy = \frac{-\alpha}{\beta}\left[e^{-\beta y}\right]_0^x$$
$$= \frac{\alpha}{\beta}(1-e^{-\beta x})$$

Hence $\frac{\alpha}{\beta}(1-e^{-\beta x}) = \frac{1}{2}u^2 - \frac{1}{2}v^2$

$$\Rightarrow \alpha - F = \beta\left(\frac{1}{2}u^2 - \frac{1}{2}v^2\right)$$
$$\Rightarrow F = \frac{1}{2}\beta v^2 + \alpha - \frac{1}{2}\beta u^2$$

When $v=0$, the force is only gravity and no air resistance. So

$$g = \alpha - \frac{1}{2}\beta u^2 \Rightarrow \alpha = \frac{1}{2}\beta u^2 + g$$

So, when moving upwards, the resistive force is $F = \frac{1}{2}\beta v^2 + g$.
Hence when moving downwards the resistive force is $F = \frac{1}{2}\beta v^2 - g$

Taking downwards as positive,

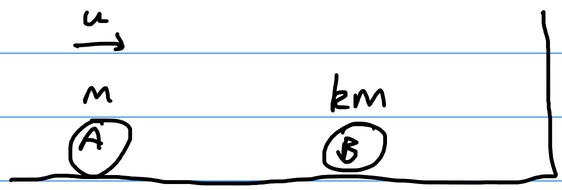
$$v\frac{dv}{dy} = g - \frac{1}{2}\beta v^2$$

$$\Rightarrow \int \frac{v}{g-\frac{1}{2}\beta v^2} dv = \int dy$$

$$\Rightarrow \frac{-1}{\beta}\ln\left|g - \frac{1}{2}\beta v^2\right| = y + c$$
$$\Rightarrow g - \frac{1}{2}\beta v^2 = Ae^{-\beta y}$$
$$\Rightarrow \frac{1}{2}\beta v^2 = g - Ae^{-\beta y}$$

$y=0 \Rightarrow$ air resistance $=0$ (as $v=0$), so $g = A$.
So air resistance is $\frac{1}{2}\beta v^2 = g(1-e^{-\beta y})$

STEP II 2008 Q10



Firs Collision:

CoM: $mu = mV_A + kmV_B \Rightarrow u = V_A + kV_B$

e: $e_1 u = V_B - V_A$ $\qquad$ $\underline{e_1 u = V_B - V_A}$

$$(1+e_1)u = (1+k)V_B$$

$$V_B = \frac{1+e_1}{1+k}u$$

$$V_A = u\left(1-\frac{k(1+e_1)}{1+k}\right)$$

$$= u\left(\frac{1+k-k-e_1k}{1+k}\right)$$

$$= \frac{1-e_1k}{1+k}u$$

After B collides with the wall, $V_B = -e_2\left(\frac{1+e_1}{1+k}\right)u$

So, for a second collision, $-e_2\left(\frac{1+e_1}{1+k}\right)u < u\frac{1-e_1k}{1+k}$

$$\Rightarrow -e_2(1+e_1) < 1-e_1k$$

$$\Rightarrow k < \frac{1+e_2(1+e_1)}{e_1}$$

Now $e_1=\frac{1}{3}$, $e_2=\frac{1}{2}$, $k<5$.

So $V_A = \frac{1-\frac{1}{3}k}{1+k}u$, $V_B = -\frac{1}{2}\left(\frac{4/3}{1+k}\right)u$

So KE $-\frac{mu^2}{27} = \frac{1}{2}mu^2\left[\left(\frac{1-\frac{1}{3}k}{1+k}\right)^2 + \frac{k}{4}\left(\frac{4/3}{1+k}\right)^2 - \frac{2}{27}\right]$

$$= \frac{1}{18}mu^2\left[\left(\frac{3-k}{1+k}\right)^2 + \frac{k}{4}\left(\frac{4}{1+k}\right)^2 - \frac{18}{27}\right]$$

$$= \frac{1}{18}mu^2\left[\frac{9-6k+k^2+4k}{(1+k)^2} - \frac{2(1+k)^2}{3(1+k)^2}\right]$$

$$= \frac{1}{54}mu^2\left[\frac{27-18k+3k^2+12k-2-4k-2k^2}{(1+k)^2}\right]$$

$$= \frac{1}{54}mu^2\left[\frac{k^2-10k+25}{(1+k)^2}\right]$$

$$= \frac{mu^2(k-5)^2}{54(1+k)^2} > 0 \text{ for } k < 5.$$

STEP II 2001 Q11

Vertically, $0 = u^2 - 2gs$ $\quad t = \frac{v-u}{a} = \frac{v}{g}$

$\Rightarrow s = \frac{v^2}{2g}$.

So the explosion occurs a distance $\frac{v^2}{2g}$ from the ground, at $t = v/g$

After this, for P,

s	$-v^2/2g$
u	V_e
v	$\times$
a	$-g$
t	?

$$\frac{-v^2}{2g} = V_e t - \frac{1}{2}gt^2$$

$$\Rightarrow \frac{1}{2}gt^2 - V_e t - \frac{v^2}{2g} = 0$$

$$\Rightarrow t = \frac{V_e \pm \sqrt{V_e^2 + v^2}}{g}$$

So $R = \left(\frac{V_e + \sqrt{V_e^2 + v^2}}{g} + \frac{v}{g}\right)u$

$$\Rightarrow \frac{gR}{u} = v + V_e + \sqrt{V_e^2 + v^2}$$

Now we require $\frac{u}{g}\left(v + V_e + \sqrt{V_e^2 + v^2}\right) > D$

$$\Rightarrow v + V_e + \sqrt{V_e^2 + v^2} > \frac{Dg}{u}$$

$$\Rightarrow V_e^2 + v^2 > \left(\frac{Dg}{u} - v - V_e\right)^2$$

$$\Rightarrow \cancel{V_e^2} + \cancel{v^2} > \frac{g^2D^2}{u^2} + \cancel{v^2} + \cancel{V_e^2} - 2\frac{vgD}{u} - 2\frac{V_e gD}{u} + 2vV_e$$

$$\Rightarrow 2V_e\left(v - \frac{gD}{u}\right) < 2\frac{vgD}{u} - \frac{g^2D^2}{u^2}$$

$$= \frac{gD}{u}\left(2v - \frac{gD}{u}\right)$$

Note $D > 2uv/g \Rightarrow \frac{gD}{u} > 2v$

$$\Rightarrow v - \frac{gD}{u} < 0,$$

so $2V_e > \frac{gD}{u} \cdot \frac{2v - gD/u}{v - gD/u}$

$$= \frac{gD}{u} \cdot \frac{2uv - gD}{uv - gD}$$

$$\Rightarrow V_e > \frac{gD}{2u}\left(\frac{gD - 2uv}{gD - uv}\right)$$

STEP II 2001 Q12

The number of non-syndicate winning tickets $X \sim B(2.8N, \frac{1}{N})$.

Note $\dfrac{P(X=k+1)}{P(X=k)} = \dfrac{(2.8N)!}{(k+1)!(2.8N-k-1)!} \cdot \left(\frac{1}{N}\right)^{k+1} \left(\frac{N-1}{N}\right)^{2.8N-k-1} \cdot \dfrac{k!(2.8N-k)!}{(2.8N)!} \cdot N^k \cdot \left(\frac{N}{N-1}\right)^{2.8N-k}$

$$= \frac{2.8N-k}{k+1} \cdot \frac{1}{N-1}$$

Setting $f(k) = \dfrac{2.8N-k}{k+1} \cdot \dfrac{1}{N-1}$,

$$f'(k) = \frac{1}{N-1} \cdot \frac{-(k+1)-(2.8N-k)}{(k+1)^2}$$

$= \dfrac{1-2.8N}{(k+1)^2(N-1)} < 0$ so $f(k)$ is decreasing in k.

$f(k) = 1 \Rightarrow 2.8N - k = (N-1)(k+1)$

$\Rightarrow 2.8N - k = Nk + N - k - 1$

$\Rightarrow k = 1.8 + \frac{1}{N} < 2$ for N large.

So, $\dfrac{P(X=k+1)}{P(X=k)} > 1$ for $k<2$, <1 for $k \geq 2$.

Hence $P(X=2)$ is maximal. Hence the most probable number of winning tickets is 3 (including the syndicate's ticket).

Now, the syndicate's expected loss is

$$N - W\sum_{k=0}^{2.8N} \binom{2.8N}{k} \left(\frac{1}{N}\right)^k \left(\frac{N-1}{N}\right)^{2.8N-k} \cdot \frac{1}{k+1}$$

For N large, we can approximate $X \sim B(2.8N, \frac{1}{N})$ with $Y \sim Po(2.8)$, so expected loss $\approx$

$$N - W\sum_{k=0}^{2.8N} e^{-2.8} \cdot \frac{2.8^k}{k!} \cdot \frac{1}{k+1}$$

$$\approx N - W\sum_{k=0}^{\infty} e^{-2.8} \cdot \frac{2.8^k}{(k+1)!}$$

$$= N - \frac{We^{-2.8}}{2.8} \sum_{k=0}^{\infty} \frac{2.8^{k+1}}{(k+1)!}$$

$$= N - \frac{5e^{-2.8}}{14} \cdot \left(\sum_{k=0}^{\infty} \frac{2.8^{k}}{k!} - 1 \right) W$$

$$= N - \frac{5e^{-2.8}}{14} \left(e^{2.8} - 1 \right) W$$

$$= N - \frac{5(1-e^{-2.8})}{14} W$$

STEP II 2001 Q13

$$\frac{1}{(1+kM)^{\alpha}} = \frac{1}{2} \Rightarrow (1+Mk)^{\alpha} = 2$$
$$\Rightarrow 1+Mk = 2^{1/\alpha}$$
$$\Rightarrow M = \frac{2^{1/\alpha}-1}{k}$$

CDF $= 1-(1+kt)^{\alpha}$

PDF $= \alpha k(1+kt)^{-\alpha-1}$

Mean is $\int_0^{\infty} \alpha k(1+kt)^{-\alpha-1} \cdot t \, dt$

$u \quad t \qquad v' \quad \alpha k(1+kt)^{-\alpha-1}$

$u' \quad 1 \qquad v \quad -(1+kt)^{-\alpha}$

$$= \left[-t(1+kt)^{-\alpha}\right]_0^{\infty} + \int_0^{\infty}(1+kt)^{-\alpha}\,dt$$

$$= 0 - \frac{1}{k(\alpha-1)}\left[(1+kt)^{-\alpha+1}\right]_0^{\infty}$$

$$= \frac{1}{k(\alpha-1)}$$

Mean is 2400 $\Rightarrow \frac{1}{k(\alpha-1)} = 2400$

So, $k = \frac{1}{2400(\alpha-1)}$

Median is 1000 $\Rightarrow \frac{2^{1/\alpha}-1}{k} = 1000$

$$\Rightarrow (2^{1/\alpha}-1)(2400(\alpha-1)) = 1000$$
$$\Rightarrow (2^{1/\alpha}-1)(\alpha-1) = \frac{1}{2.4}$$

Note $\frac{1}{2.4} \approx \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$

$= (2^{1/2}-1)(2-1)$

So $\alpha \approx 2$. Then $k \approx \frac{1}{2400}$

So $P(T>m \mid T>M) = \frac{P(T>m)}{P(T>M)}$

$$= \frac{\left(1 + k \cdot \frac{1}{k(\alpha - 1)}\right)^{-\alpha}}{1/2}$$

$$= 2 \cdot \left(1 + \frac{1}{\alpha - 1}\right)^{-2}$$

$$= 2 \cdot 2^{-2}$$

$$= \frac{1}{2}$$

STEP II 2001

$$T = \lambda X + \tfrac{1}{2}(1-\lambda)Y$$

$$\begin{aligned} ET &= \lambda EX + \tfrac{1}{2}(1-\lambda)EY \\ &= \lambda p + \tfrac{1}{2}(1-\lambda)2p \\ &= \lambda p + \tfrac{1}{2} - \lambda p \\ &= p \end{aligned}$$

$$\begin{aligned} \text{Var}\,T &= \lambda^2 \text{Var}\,X + \tfrac{1}{4}(1-\lambda)^2 \text{Var}\,Y \\ &= \lambda^2 p(1-p) + \tfrac{1}{4}(1-\lambda)^2 2p(1-2p) \\ &= \lambda^2 p(1-p) + \tfrac{1}{2}(1-\lambda)^2 p(1-2p) \end{aligned}$$

$$\frac{d}{d\lambda} = 2\lambda p(1-p) - (1-\lambda)p(1-2p) = 0$$

$$\Rightarrow 2\lambda(1-p) + \lambda(1-2p) = 1-2p$$

$$\Rightarrow \lambda(2-2p+1-2p) = 1-2p$$

$$\Rightarrow \lambda = \frac{1-2p}{3-4p}$$

$$\begin{aligned} \frac{d^2}{d\lambda^2} &= 2p(1-p) + p(1-2p) \\ &= 2p - 2p^2 + p - 2p^2 \\ &= p(3-4p) > 0 \text{ as } p < \tfrac{1}{2}, \text{ so } \lambda = \tfrac{1-2p}{3-4p} \text{ is a minimum.} \end{aligned}$$

$$\begin{aligned} \text{Then Var}\,T &= \lambda^2 p(1-p) + \tfrac{1}{4}(1-\lambda)^2 2p(1-2p) \\ &= \frac{(1-2p)^2}{(3-4p)^2} p(1-p) + \frac{1}{2}\frac{(2-2p)^2}{(3-4p)^2} p(1-2p) \\ &= \frac{p(1-p)(1-2p)}{(3-4p)^2}\left[1-2p+2(1-p)\right] \\ &= \frac{p(1-p)(1-2p)}{3-4p} \end{aligned}$$

So $\bar{T} \overset{\text{approx}}{\sim} N\left(p, \frac{p(1-p)(1-2p)}{n(3-4p)}\right)$ by CLT

so $P(|\bar{T}-p|<b) = \Phi\left(\frac{b}{s}\right) - \Phi\left(\frac{-b}{s}\right)$

$= \Phi\left(\frac{b}{s}\right) - \left(1-\Phi\left(\frac{b}{s}\right)\right)$

$= 2\Phi\left(\frac{b}{s}\right) - 1$