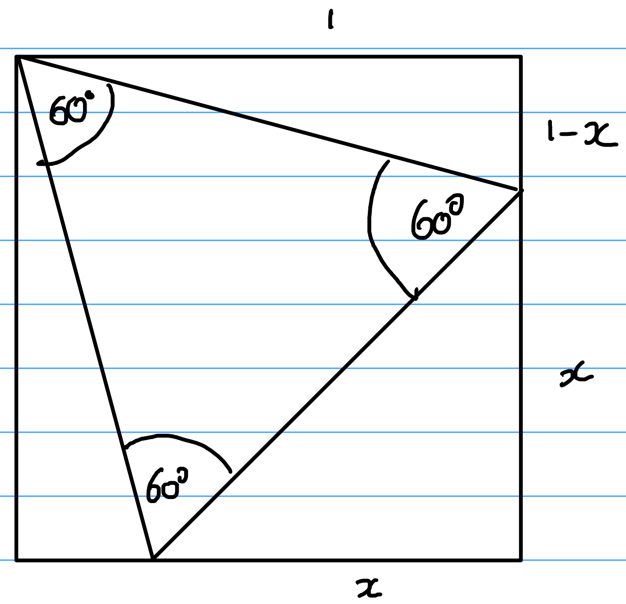
STEP I 2001 Q1

We want to maximise the shortest side of the triangle. We can always make the shortest side longer by moving one of its vertices slightly further away from the other. This will increase the length of the shortest side unless another side becomes shorter under this vertex shift. So, keep making the shortest side longer until it is no longer possible – this will be when all three sides are the same length. So we have an equilateral triangle. Clearly to maximise side length, one of the vertices will be at a vertex of the square.



Because the side lengths of the triangle are the same, by Pythagoras' Theorem we have

$$1+(1-x)^2 = x^2+x^2$$
$$\Rightarrow 2-2x+x^2 = 2x^2$$
$$\Rightarrow x^2+2x-2=0$$
$$\Rightarrow x = \frac{-2+\sqrt{4+8}}{2}$$
$$= -1+\sqrt{3}$$

Then side length of triangle is $\sqrt{2(-1+\sqrt{3})^2}$

$$= \sqrt{2}(-1+\sqrt{3})$$
$$= \sqrt{6}-\sqrt{2}, \text{ as required.}$$

STEP I 2001 Q2

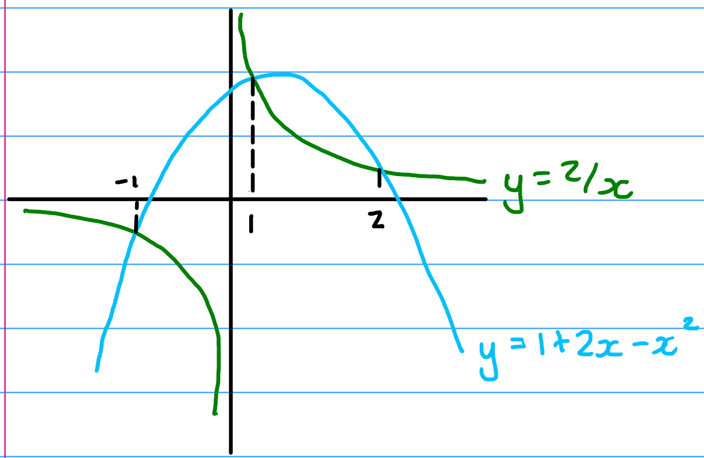
(i) Suppose $1 + 2x - x^2 > 2/x$

$\Rightarrow x^3 - 2x^2 - x + 2 = 0$

Note $f(1) = f(-1) = f(2) = 0$, so

$= (x-2)(x+1)(x-1)$

Hence the curves intersect at $x=-1, x=1, x=2$



So $1+2x-x^2 > 2/x$ has solution

$-1 < x < 0$ or $1 < x < 2$

(ii) Suppose $\sqrt{3x+10} = 2 + \sqrt{x+4}$

$\Rightarrow 3x+10 = 4 + 4\sqrt{x+4} + x + 4$

$\Rightarrow 2x+2 = 4\sqrt{x+4}$

$\Rightarrow (x+1)^2 = 4(x+4)$

$\Rightarrow x^2+2x+1 = 4x+16$

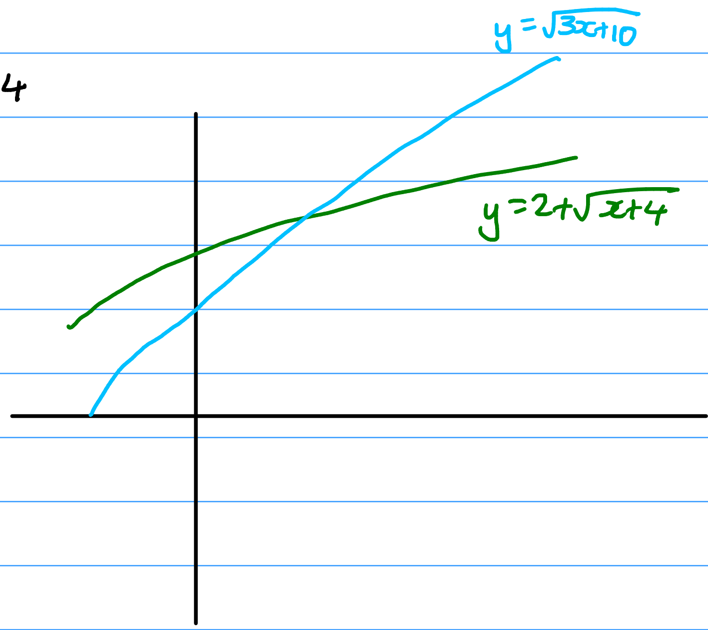
$\Rightarrow x^2-2x-15=0$

$\Rightarrow (x-5)(x+3)=0$

$\Rightarrow x=5$ or $x=-3$

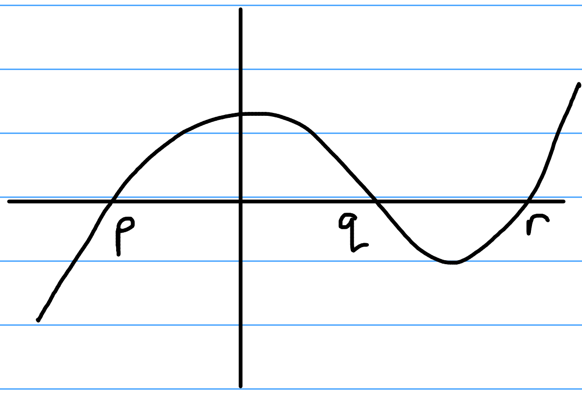
Note $\sqrt{3\times5+10} = 2+\sqrt{5+4} = 5$

but $1 = \sqrt{3\times-3+10} \neq 2+\sqrt{-3+4} = 3$



Hence the solution is $x > 5$

STEP I 2001 Q3



Clearly $y=f(x)$ has two stationary points, so $f'(x)=0$ has two solutions.

$$f(x) = x^3-(p+q+r)x^2+(pq+qr+rp)x-pqr$$
$$f'(x) = 3x^2-2(p+q+r)x+(pq+qr+rp)$$

$b^2-4ac>0$

$$\Rightarrow \quad 4(p+q+r)^2-12(pq+qr+rp)>0$$
$$\Rightarrow \quad (p+q+r)^2>3(pq+qr+rp)$$

Now $f(x)=(x^2+gx+h)(x-k)$

If $g^2>4h$, then x^2+gx+h has two distinct real roots α and β with $\alpha+\beta=-g$, $\alpha\beta=h$. Then by the above,

$$(\alpha+\beta+k)^2>3(\alpha\beta+(\alpha+\beta)k)$$
$$\Rightarrow (-g+k)^2>3(h-gk)$$
$$\Rightarrow (g-k)^2>3(h-gk)$$

So $g^2>4h$ is sufficient.

But now suppose $g=h=0$, $k=1$. Then clearly $g^2>4h$ is not true, but

$$(g-k)^2=1, \quad 3(h-gk)=0, \quad \text{so } (g-k)^2>3(h-gk).$$

So $g^2>4h$ is not necessary.

STEP I 2001 Q4

$$\tan 3\theta = \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan\theta}{1 - \tan\theta \tan 2\theta}$$

$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \tan\theta \cdot \frac{2\tan\theta}{1-\tan^2\theta}}$$

$$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$\theta = \cos^{-1}(2/\sqrt{5})$, $0 < \theta < \pi/2$ $\Rightarrow$ $\cos\theta = 2/\sqrt{5}$

$\sin\theta = 1/\sqrt{5}$

$\tan\theta = 1/2$

$$\Rightarrow \tan 3\theta = \frac{3(1/2) - (1/8)}{1 - 3(1/4)}$$

$$= \frac{12 - 1}{8 - 6}$$

$$= 11/2$$

i) Suppose $\tan(3\cos^{-1}(x)) = 11/2$

Let $\theta = \cos^{-1}(x) \Rightarrow \tan 3\theta = 11/2$

$\Rightarrow \theta = \cos^{-1}(2/\sqrt{5})$

$\Rightarrow x = 2/\sqrt{5}$

But $3\cos^{-1}(x) \in [0, 3\pi]$, so we also need two more solutions.

$3\cos^{-1}x = 3\cos^{-1}(2/\sqrt{5}) + \pi$

$\Rightarrow \cos^{-1}x = \cos^{-1}(2/\sqrt{5}) + \pi/3$

$\Rightarrow x = \cos\left(\cos^{-1}(2/\sqrt{5}) + \pi/3\right)$

$= 2/\sqrt{5}\cos\pi/3 - \sin(\cos^{-1}(2/\sqrt{5}))\sin\pi/3$

$$= \frac{2}{\sqrt5}\cdot\frac{1}{2} - \frac{1}{\sqrt5}\cdot\frac{\sqrt3}{2}$$
$$= \frac{1}{\sqrt5}\left(1-\frac{\sqrt3}{2}\right)$$

Or, $3\cos^{-1}x = 3\cos^{-1}(2/\sqrt5) + 2\pi$

$$\Rightarrow \cos^{-1}x = \cos^{-1}(2/\sqrt5) + 2\pi/3$$
$$\Rightarrow x = \frac{2}{\sqrt5}\cos\frac{2\pi}{3} - \frac{1}{\sqrt5}\sin\frac{2\pi}{3}$$
$$= \frac{2}{\sqrt5}(-1/2) - \frac{1}{\sqrt5}(\sqrt3/2)$$
$$= \frac{-1}{\sqrt5}(1+\sqrt3/2)$$

So, $x = 2/\sqrt5,\ \frac{1}{\sqrt5}(1-\sqrt3/2),\ -\frac{1}{\sqrt5}(1+\sqrt3/2)$

l.i) $\cos(\frac{1}{3}\tan^{-1}y) = 2/\sqrt5$ Note $\frac{1}{3}\tan^{-1}y \in [-\pi/6, \pi/6]$, so there will be two (or zero) solutions, each the negative of each other.

Suppose $\frac{1}{3}\tan^{-1}y = \cos^{-1}(2/\sqrt5)$

$$\Rightarrow y = \tan(3\cos^{-1}(2/\sqrt5))$$
$$= 11/2$$

So $y = 11/2$ or $y = -11/2$

STEP I 2001 Q5

(i) $\displaystyle\int_0^1 \frac{1}{(1+tx)^2}\,dx = \left[\frac{-1}{t(1+tx)}\right]_0^1$

$$= \frac{-1}{t(1+t)} + \frac{1}{t}$$

$$= \frac{-1+1+t}{t(1+t)}$$

$$= \frac{1}{(1+t)}$$

(ii) $\displaystyle\int_0^1 \frac{-2x}{(1+tx)^3}\,dx$ Integrate by parts, with

u	$-2x$	v'	$(1+tx)^{-3}$
u'	-2	v	$\frac{-1}{2t}(1+tx)^{-2}$

$$= \left[\frac{2x}{2t(1+tx)^2}\right]_0^1 - \int_0^1 \frac{1}{t(1+tx)^2}\,dx$$

$$= \frac{1}{t(1+t)^2} - \frac{1}{t}\cdot\frac{1}{1+t}$$

$$= \frac{1-1-t}{t(1+t)^2}$$

$$= \frac{-1}{(1+t)^2}$$

Note that $\displaystyle\frac{d}{dt}\frac{1}{(1+tx)^2} = \frac{-2x}{(1+tx)^3}$

$$\frac{d}{dt}\left(\frac{-2x}{(1+tx)^3}\right) = \frac{6x^2}{(1+tx)^4}$$

So we conjecture that $\displaystyle\int_0^1 \frac{6x^2}{(1+tx)^4}\,dx = \frac{d}{dt}\frac{-1}{(1+t)^2}$

$$= \frac{2}{(1+t)^3}$$

Setting $t=1$ gives $\displaystyle\int_0^1 \frac{6x^2}{(1+x)^4}\,dx = \frac{2}{8}$

$$= \frac{1}{4}$$

STEP I 2001 Q6

Consider the curve $y=\sqrt{r^2-x^2}$, then $\frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$.

Suppose a fraction α of the bread is left. Then the remaining surface area is

$$2\pi\int_{-r}^{-r+\alpha(2r)} \sqrt{r^2-x^2}\left(1+\frac{x^2}{r^2-x^2}\right)^{1/2} dx$$

$$= 2\pi\int_{-r}^{-r+2\alpha r} (r^2-x^2+x^2)^{1/2}\, dx$$

$$= 2\pi r\int_{-r}^{-r+2\alpha r} dx$$

$$= 2\pi r(-r+2\alpha r+r)$$

$$= 4\pi r^2 \times \alpha.$$

The surface area of the whole loaf is $4\pi r^2$, so the remaining surface area is proportional to the number of slices remaining.

The fraction of volume remaining is

$$\pi\int_{-r}^{-r+2\alpha r} r^2-x^2\, dx$$

$$=\pi\left[r^2x-\tfrac{1}{3}x^3\right]_{-r}^{-r+2\alpha r}$$

$$= \pi r^3\left[(-1+2\alpha)-\tfrac{1}{3}(-1+2\alpha)^3+1-\tfrac{1}{3}\right]$$

$$= \pi r^3\left[-1+2\alpha-\tfrac{1}{3}(-1+6\alpha-12\alpha^2+8\alpha^3)+1-\tfrac{1}{3}\right]$$

$$=\pi r^3\left(-1+2\alpha+\tfrac{1}{3}-2\alpha+4\alpha^2-\tfrac{8}{3}\alpha^3+1-\tfrac{1}{3}\right)$$

$$=\pi r^3\left(4\alpha^2-\tfrac{8}{3}\alpha^3\right)$$

Since we must have $V/A<1$, we require

$$\frac{\pi r^3 \alpha^2 (4 - \frac{8}{3}\alpha)}{4\pi r^2 \alpha} < 1 \qquad \text{for all } \alpha \in [0,1]$$

$$\Rightarrow \frac{r}{4}\left(4\alpha - \frac{8}{3}\alpha^2\right) < 1$$

Considering $4\alpha - \frac{8}{3}\alpha^2$, the derivative is $4 - \frac{16}{3}\alpha = 0$

$$\Rightarrow \alpha = \frac{3}{4}$$

So the maximum value of $4\alpha - \frac{8}{3}\alpha^2$ is $4\left(\frac{3}{4}\right) - \frac{8}{3}\left(\frac{3}{4}\right)^2$

$$= 3 - \frac{3}{2}$$

$$= \frac{3}{2}$$

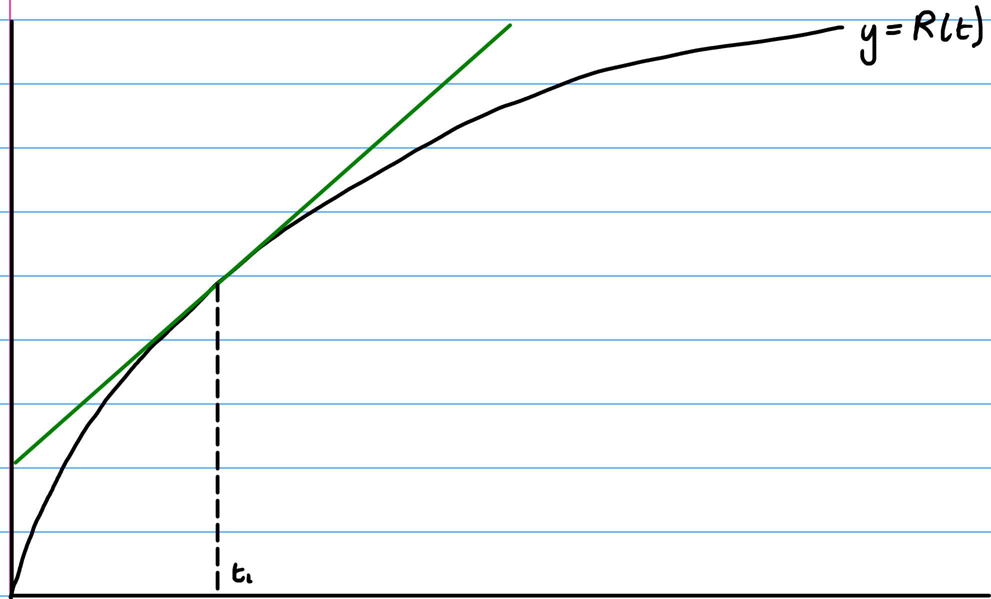
So we have $\frac{r}{4} \times \frac{3}{2} < 1$

$\Rightarrow r < \frac{8}{3} = 2\frac{2}{3}$, as required.

STEP I 2001 Q7

$R(0)=0, \quad R'(t)>0 \quad R''(t)<0$

(i)



The equation of the tangent is

$$y - R(t_1) = R'(t_1)(t-t_1)$$

$$\Rightarrow y = R'(t_1)t + R(t_1) - t_1 R'(t_1)$$

The y-intercept of the tangent is positive, so

$$R(t_1) - t_1 R'(t_1) > 0$$

$$\Rightarrow t_1 < \frac{R(t_1)}{R'(t_1)} \qquad (\text{since } R'(t_1) > 0)$$

$$\Rightarrow t < \frac{1}{H(t)} \quad \text{for all } t>0.$$

(ii) $\dfrac{R'(t)}{R(t)} = \dfrac{a}{t}$

$\Rightarrow \ln R(t) = a\ln t + c$

$\Rightarrow \quad R(t) = k \times t^a$

$R(0)=0$ ✓ $\quad R(t) \geqslant 0$ for all $t>0 \Rightarrow k>0$

$R'(t) = kat^{a-1} \quad > 0 \quad \Rightarrow \quad a>0$

$R''(t) = ka(a-1)t^{a-2} < 0 \quad \Rightarrow \quad 0<a<1$

So we require $0<a<1$.

(iii) Now, $\frac{R'(t)}{R(t)} = \frac{b}{t^2}$

$$\Rightarrow \ln R(t) = -b/t + c$$
$$\Rightarrow R(t) = ke^{-b/t}$$

As before, $k>0$.

$$\lim_{t\to 0} R(t) = \begin{cases} 0 & b>0 \\ \infty & b<0 \end{cases}$$

so $R(0) = 0 \Rightarrow b>0$

$$R'(t) = \frac{kb}{t^2} e^{-b/t} > 0$$

$$R''(t) = e^{-b/t}\left(\frac{kb^2}{t^4} - \frac{2kb}{t^3}\right)$$

$$= \frac{kb}{t^3} e^{-b/t}\left(\frac{b}{t} - 2\right)$$

For small values of t $(0<t<\frac{b}{2})$, $R''(t)>0$, violating the conditions.

STEP I 2001 Q8

$y'' + p(x)y' + q(x)y = 0$

$y = x \Rightarrow \quad p(x) + xq(x) = 0$

$\Rightarrow p(x) = -xq(x)$

$y = 1-x^2 \Rightarrow -2 - 2xp(x) + (1-x^2)q(x) = 0$

$\Rightarrow -2 + 2x^2q(x) + (1-x^2)q(x) = 0$

$\Rightarrow q(x) = \frac{2}{1+x^2}$

$\Rightarrow p(x) = \frac{-2x}{1+x^2}$

So $y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 0$

If $y = ax + b(1-x^2)$, then this is

$$-2b - \frac{2x}{1+x^2}(a-2bx) + \frac{2}{1+x^2}(ax+b-bx^2)$$

$$= b\left(-2 + \frac{4x^2}{1+x^2} + \frac{2}{1+x^2} - \frac{2x^2}{1+x^2}\right) + a\left(\frac{-2x}{1+x^2} + \frac{2x}{1+x^2}\right)$$

$$= \frac{b}{1+x^2}(-2-2x^2+4x^2+2-2x^2) + 0$$

$= 0$, as required.

We must now also have $a\sin^2\frac{x^2}{2} + b\cos^2\frac{x^2}{2}$ as a solution. Setting $a=b=1$ means $y = \sin^2\frac{x^2}{2} + \cos^2\frac{x^2}{2} = 1$ satisfies the equation

$\Rightarrow \quad 0 + 0 \times p(x) + q(x) = 0$

$\Rightarrow q(x) = 0$

Setting $a=-1, b=1$ gives $y = \cos(x^2)$.

Then $y' = -2x\sin x^2$

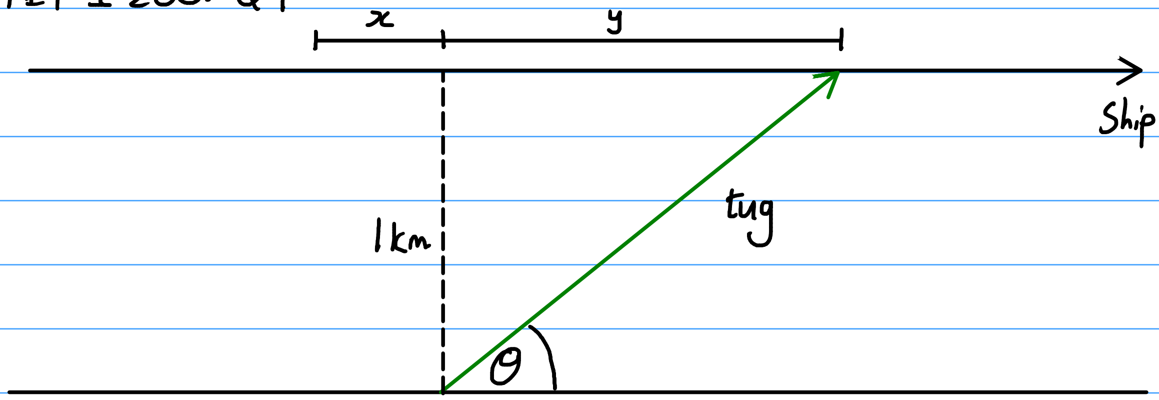
$y'' = -2\sin x^2 - 4x^2\cos x^2$

Substituting in, $-2\sin x^2 - 4x^2\cos x^2 - p(x)\,2x\sin x^2 = 0$

$$\Rightarrow p(x) = -\frac{\sin x^2 + 2x^2\cos x^2}{x\sin x^2}$$

$$= -\frac{1}{x} - 2x\cot x^2$$

STEP I 2001 Q9



Suppose the tug leaves when the ship is x km from the point of closest approach. Then

$$\underbrace{\frac{x+y}{20}}_{\text{time for boat}} = \underbrace{\frac{y}{12\cos\theta}}_{\text{time for tug}} = \underbrace{\frac{1}{12\sin\theta}}_{\text{time for tug}}$$

$$\frac{y}{12\cos\theta} = \frac{1}{12\sin\theta} \Rightarrow y = \cot\theta$$

So, $$\frac{x+\cot\theta}{20} = \frac{\cot\theta}{12\cos\theta}$$

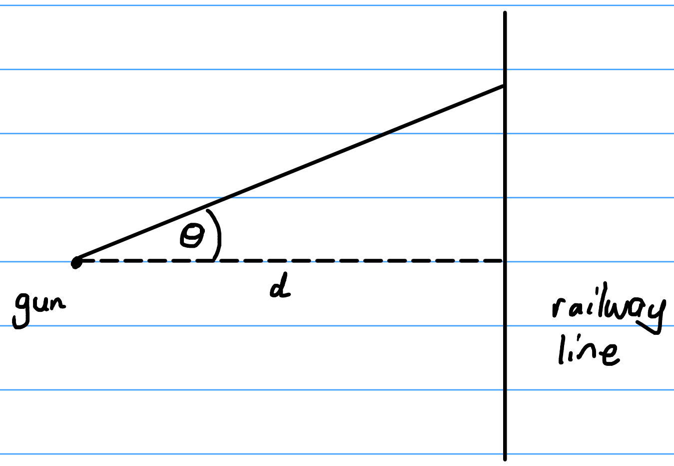
$$\Rightarrow x + \cot\theta = \frac{5}{3}\operatorname{cosec}\theta$$
$$\Rightarrow x = \frac{5}{3}\operatorname{cosec}\theta - \cot\theta$$

We want to minimise x over θ, so

$$\frac{dx}{d\theta} = -\frac{5}{3}\operatorname{cosec}\theta\cot\theta - \operatorname{cosec}^2\theta = 0$$
$$\Rightarrow -\frac{5}{3}\cot\theta + \operatorname{cosec}\theta = 0$$
$$\Rightarrow -\frac{5}{3}\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = 0$$
$$\Rightarrow \cos\theta = \frac{3}{5}$$

The distance travelled by the tug is $\operatorname{cosec}\theta = \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{1-9/25}}$
$= 1.25\text{km}$

STEP I 2001 Q10



Suppose the gunner aims an angle θ from the closest point on the railway line, so the horizontal distance to travel is $\frac{d}{\cos\theta}$. If the angle of elevation of the shot is φ, then horizontally,

$$T = \frac{d/\cos\theta}{v\cos\varphi} = \frac{d}{v\cos\theta\cos\varphi} \qquad (1)$$

vertically,

$$0 = v\sin\varphi T - \frac{1}{2}gT^2 \qquad (2)$$

where T is the time of flight.

Because $T \neq 0$, (2) gives $T = \frac{2v\sin\varphi}{g}$

$$\Rightarrow \sin\varphi = \frac{gT}{2v}$$

(1) gives $\cos\varphi = \frac{d}{vT\cos\theta}$

Then using $\sin^2\varphi + \cos^2\varphi = 1$, we have

$$\frac{g^2T^2}{4v^2} + \frac{d^2}{v^2T^2\cos^2\theta} = 1$$

$$\Rightarrow g^2T^4 + \frac{4d^2}{\cos^2\theta} = 4v^2T^2$$

$$\Rightarrow g^2(T^2)^2 - 4v^2T^2 + \frac{4d^2}{\cos^2\theta} = 0$$

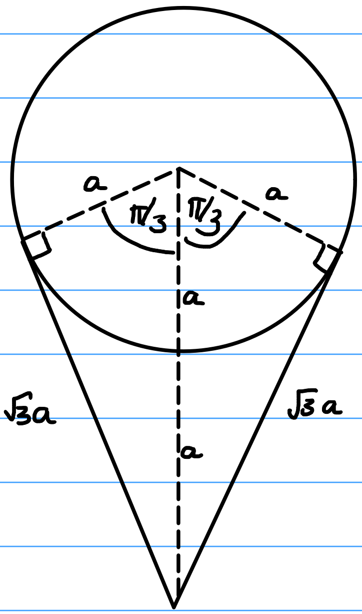
$$\Rightarrow T^2 = \frac{4v^2 \pm \sqrt{16v^4 - 16d^2g^2/\cos^2\theta}}{2g^2}$$

$$\Rightarrow g^2T^2 = 2v^2 \pm 2\sqrt{v^4 - d^2g^2/\cos^2\theta}$$

We want to maximise T, so choose the positive square root and maximise $\cos^2\theta$
$\Rightarrow \cos^2\theta = 1$.

Then $g^2T^2 = 2v^2 + 2\sqrt{v^4 - g^2d^2}$

STEP I 2001 Q11



(i) The length of the band is $\frac{4\pi}{3}a + 2\sqrt{3}a$

$\Rightarrow$ extension $= a(2\sqrt{3} - \frac{2\pi}{3})$

$$EPE = \frac{\lambda x^2}{2l} = \frac{\lambda a^2(2\sqrt{3} - \frac{2\pi}{3})^2}{4\pi a}$$

$$= \frac{\lambda a(\sqrt{3} - \pi/3)^2}{\pi}$$

$GPE = mga$

So, $mga = \dfrac{\lambda a(\sqrt{3} - \pi/3)^2}{\pi}$

$$\Rightarrow \lambda = \frac{mg\pi}{(\sqrt{3} - \pi/3)^2}$$

$$= \frac{9mg\pi}{(3\sqrt{3} - \pi)^2}$$

(ii) acceleration $= 0 \Rightarrow$ resultant force $= 0$

$\Rightarrow 2T\cos\frac{\pi}{6} = mg$

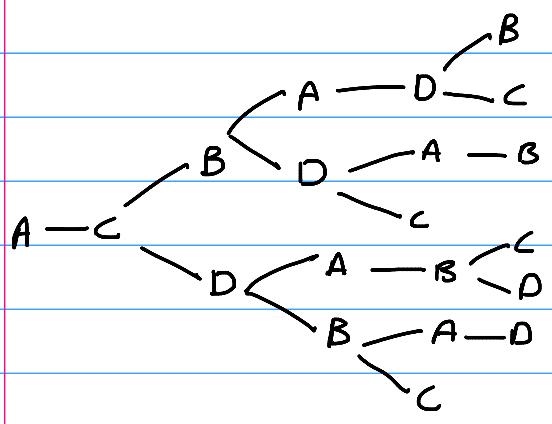
$$T = \frac{\lambda x}{l} = \frac{\lambda(2\sqrt{3} - 2\pi/3)\cancel{a}}{2\pi \cancel{a}}$$

so $\dfrac{\cancel{2}\lambda(2\sqrt{3} - 2\pi/3)}{\cancel{2}\pi} \cdot \dfrac{\sqrt{3}}{2} = mg$

$$\Rightarrow \lambda = \frac{2mg\pi}{\sqrt{3}(2\sqrt{3} - 2\pi/3)}$$

$$= \frac{3mg\pi}{9 - \pi\sqrt{3}}$$

STEP I 2009 Q12



Each branch has terminal probability $\frac{1}{8}$. There are 6 in which A appears twice so

$P(A \text{ rehears}) = 6/8 = 3/4$

$P(B \text{ rehears}) = 2/8 = 1/4$

$P(C \text{ rehears}) = 4/8 = 1/2$

STEP I 2001 Q13

$P(M \text{ breaks} \geq 3 \mid 5 \text{ broken})$

$$= \frac{P(5 \text{ broken} \cap M \text{ breaks} \geq 3)}{P(5 \text{ broken})}$$

$$= \frac{P(M \text{ breaks } 3, \text{ others } 2) + P(M \text{ breaks } 4, \text{ others } 1) + P(M \text{ breaks } 5, \text{ others } 0)}{P(5 \text{ broken})}$$

$$= \frac{\frac{\cancel{e^{-\lambda}}\cancel{\lambda^3}}{3!}\frac{\cancel{e^{-3\lambda}}(3\cancel{\lambda})^2}{2!} + \frac{\cancel{e^{-\lambda}}\cancel{\lambda^4}}{4!}\cdot\cancel{e^{-3\lambda}}(3\cancel{\lambda}) + \frac{\cancel{e^{-\lambda}}\cancel{\lambda^5}}{5!}\cdot\cancel{e^{-3\lambda}}}{\frac{\cancel{e^{-4\lambda}}(4\cancel{\lambda})^5}{5!}}$$

$$= \frac{\frac{9}{6\times 2} + \frac{3}{24} + \frac{1}{120}}{1024/120}$$

$$= \frac{90+15+1}{1024}$$

$$= \frac{106}{1024}$$

$$= \frac{53}{512}$$

STEP I 2001 Q14

(i) We need the best candidate in the first n, with probability $\frac{n}{N}$.

(ii) The number of ways that neither of the best two candidates is in the first n is $(N-n)(N-n-1)(N-2)!$.

So, the probability that either (or both) is in the first n is

$$\frac{1}{N!}\left(N! - (N-n)(N-n-1)(N-2)!\right)$$
$$= 1 - \frac{(N-n)(N-n-1)}{N(N-1)}$$

Now with $N=4$, $n=2$. Ranking the candidates 1 to 4, with 1 being the strongest, the possible pairs of the first two interviewed is

1 2	2 1	3 1	4 1
1 3	2 3	3 2	4 2
1 4	2 4	3 4	4 3

Of these, $\frac{6}{12} = \frac{1}{2}$ contain the best candidate. The formula gives $\frac{2}{4} = \frac{1}{2}$ ✓

Of these, $\frac{10}{12} = \frac{5}{6}$ contain the best or second-best candidate. The formula gives $1 - \frac{2\times 1}{4\times 3} = 1 - \frac{1}{6} = \frac{5}{6}$ ✓