STEP II 2000 Comments

Question 1

A pretty decent first question on number theory, and a quite interesting result, too. As it turns out, any rational number can be written as the sum of (not necessarily two) distinct unit fracions. These are known as *Egyptian fractions* as the ancient Egyptians used only unit fractions (and occasionally $\frac{2}{3}$ and $\frac{3}{4}$) in their calculations. The Rhind Papyrus includes a table showing the Egyptian fraction decomposition of $\frac{2}{n}$ for $3 \le n \le 101$, n odd.

Question 2

A fairly easy start, I think, with the main difficulty coming when needing to distinguish between the three roots of $p'''(x) = 0$. Because you are told that $p(x)$ has a factor of the form $(x - a)^4$ it is enough to show that two of these roots can't work – the final root must give a factor.

Question 3

I'm not really sure what to think about this question. It's unlike other STEP questions that I have seen. Of course the vital insight is to ignore terms of quadratic or higher order in the error terms. I think the extra bit at the end to find the bounds for η is a bit pointless, and was probably only tacked on because someone thought the original question was too short!

Question 4

A fairly standard bookwork start with a proof of De Moivre's Theorem, followed by proving some results using the argument. The trick to spot is how the $(5 - i)^2$ relates to $2\arctan\frac{1}{5}$. Once that is clear, the rest should follow. The final part requires you to come up with your own expression, by using the given result and working backwards.

Question 5

This is a nice question. Differentiating the first expression with respect to λ is tricky if you haven't seen that sort of thing before (the fact that all the x terms can be treated as constant as the definite integral just turns them into a number). I was caught out working through this by using $\sin\theta \approx \theta$, rather than $\sin\theta \approx \theta - \frac{1}{6}\theta^3$.

Question 6

I think this one is a little fiddly, particularly at the start. An "obvious" way to show the third result is to use the compound angle formula for tan, but that quickly requires you to evaluate $\tan\frac{\pi}{2}$ which causes problems! Spotting how to turn the final integral into the penultimate one saves a lot of time.

Question 7

A tricky one, with good geometric intuition needed. The first part is fairly straightforward, but then working out how to apply this to the second part does require some thought.

Question 8

The first part is a reasonably standard separable differential equation, and the second part is similar but with the slight added complexity of including a parameter. Determining the value of k requires bringing the e^{-3x^2} into the bracket and noting that power in the second term must be zero so that the limit is non-zero.

Question 9

This question was a massive slog, to be honest! The partial fractions in particular were very fiddly. I also think it's a bit of a stretch having this in the mechanics section – the only mechanics involved is writing down the differential equation for Jane and Karen! After that everything is just a long algebra exercise.

Question 10

I like this question. Multiple connected pulleys are a good extension of A Level material and I like how some kinematics is also needed here to solve the second half of the question. Keeping track of both t and T is vital!

Question 11

Another good (if a little lengthy) question. None of the principles involved here are too bad, but holding your nerve to work through all the algebra to end up with the given answer is not easy!

Question 12

A nice question on combining normal distributions, but possibly a little short. Normal distributions are fairly unique that if you add them you still get a normal distribution.

Question 13

Capture-recapture methods are used in biology and ecology to estimate population sizes, and are a useful tool. They are well understood statistically. As it turns out, the maximum likelihood estimate for the population size is $\frac{nN}{k}$ where n is the size of the first sample, N the size of the second sample, and k the number of marked individuals in the second sample. In our case, $n = N = 200$ and $k = 11$ gives an estimate of $\frac{40,000}{11}$ as we found.

Question 14

The given integral result here is a (slightly shifted) beta function, a very important function in complex analysis – functions here are often defined by integrals, including the famous Riemann zeta function. I don't think this question is too tricky, although a little thought is needed for the final part.