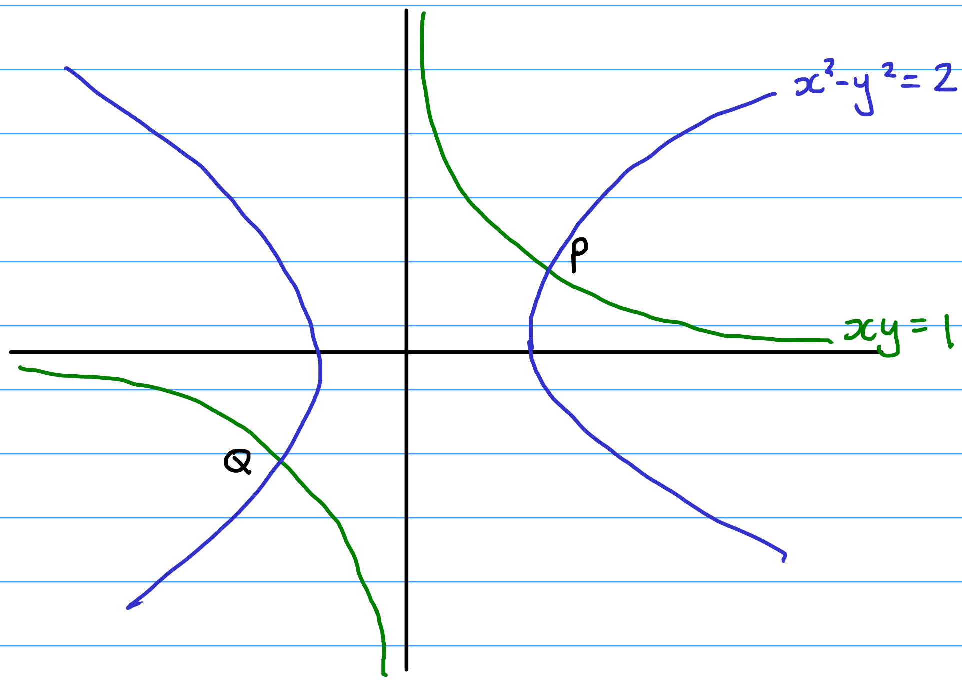
STEP III 2000 Q1



$P(a,b)$, $Q(-a,-b)$

Tangent to C_1 through P: $y = 1/x \Rightarrow y' = -\frac{1}{x^2}$

So, $y - b = \frac{-1}{a^2}(x-a)$

Normal to C_2 through Q: $x^2-y^2=2 \Rightarrow 2x - 2y\frac{dy}{dx} = 0$

$\Rightarrow dy/dx = x/y$

So, $y+b = \frac{a}{b}(x+a)$

Using $ab=1$ gives

$$y-b = -b^2(x-a)$$

and $$y+b = a^2(x+a)$$

Subtracting gives

$$2b = a^2(x+a) + b^2(x-a)$$
$$= x(a^2+b^2) + a(a^2-b^2)$$
$$= x(a^2+(a^2-2)) + 2a \qquad (\text{as } a^2-b^2=2)$$
$$= 2x(a^2-1) + 2a$$

So $b = x(a^2-1) + a$

$$\Rightarrow x = \frac{b-a}{a^2-1}$$
$$= \frac{b-1/b}{1/b^2-1}$$
$$= \frac{-b(-1+1/b^2)}{1/b^2-1}$$
$$= -b$$

Then $y = -b + a^2(x+a)$
$$= -b + a^2(-b+a)$$
$$= -b + a^2\left(\frac{-1}{a} + \frac{1}{b}\right)$$
$$= -a + \frac{1}{b}(a^2-b^2)$$
$$= -a + a(2)$$
$$= a$$

So M is $(-b, a)$ and N is $(b, -a)$.

Note that the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a $\pi/2$ rotation about the origin, and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} b \\ -a \end{pmatrix}$$
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

So each vertex is rotated by $\pi/2$ about the origin to obtain the next vertex. Hence $PMQN$ is a square.

STEP III 2000 Q2

$$\int_{3/2}^{2} \left(\frac{x-1}{3-x}\right)^{1/2} dx$$

$x = 2 - \cos\theta$

$dx = \sin\theta \, d\theta$

$$= \int_{\pi/3}^{\pi/2} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \sin\theta \, d\theta$$

$$= \int_{\pi/3}^{\pi/2} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \sqrt{\frac{1-\cos\theta}{1-\cos\theta}} \sin\theta \, d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{1-\cos\theta}{\sqrt{1-\cos^2\theta}} \sin\theta \, d\theta$$

$$= \int_{\pi/3}^{\pi/2} 1 - \cos\theta \, d\theta$$

$$= \left[\theta - \sin\theta\right]_{\pi/3}^{\pi/2}$$

$$= \left(\pi/2 - 1\right) - \left(\pi/3 - \sqrt{3}/2\right)$$

$$= \pi/6 + \sqrt{3}/2 - 1 .$$

Now consider

$$\int_{3a+b/4}^{a+b/2} \sqrt{\frac{x-a}{b-x}} \, dx$$

Consider a substitution of the form $x = \alpha - \beta\cos\theta$. Then the integrand becomes

$$\sqrt{\frac{x-a}{b-x}} = \sqrt{\frac{(\alpha-a)-\beta\cos\theta}{(b-\alpha)+\beta\cos\theta}}$$

For the same trick as earlier to work, we need $(\alpha-a)=(b-\alpha)$ and $(\alpha-a)=\beta$

So, $\alpha = \frac{a+b}{2}$ and $\beta = \frac{b-a}{2}$

So use $x = \frac{a+b}{2} - \frac{b-a}{2}\cos\theta$

For the limits, $x = \frac{3a+b}{4} \Rightarrow 3a+b = 2a+2b-(2b-2a)\cos\theta$

$$\Rightarrow a-b = 2(a-b)\cos\theta$$

$$\Rightarrow \cos\theta = 1/2$$

$$\Rightarrow \theta = \pi/3$$

$$x = \frac{a+b}{2} \Rightarrow \frac{a+b}{2} = \frac{a+b}{2} - \frac{b-a}{2}\cos\theta$$

$$\Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = \pi/2$$

And finally, $dx = \frac{b-a}{2}\sin\theta\, d\theta$

So the integral becomes

$$\int_{\pi/3}^{\pi/2} \sqrt{\frac{(b-a)/2 - (b-a)/2\cos\theta}{(b-a)/2 + (b-a)/2\cos\theta}} \cdot \left(\frac{b-a}{2}\right)\sin\theta\, d\theta$$

$$= \frac{b-a}{2}\int_{\pi/3}^{\pi/2} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta\, d\theta$$

$$= \frac{b-a}{2}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1\right)$$

$$= \frac{(b-a)(\pi+3\sqrt{3}-6)}{2}$$, as required.

STEP III 2000 Q3

$\alpha = e^{i\pi/3}$, $1+\alpha^2 = 1+e^{2\pi i/3}$

$= 1+\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

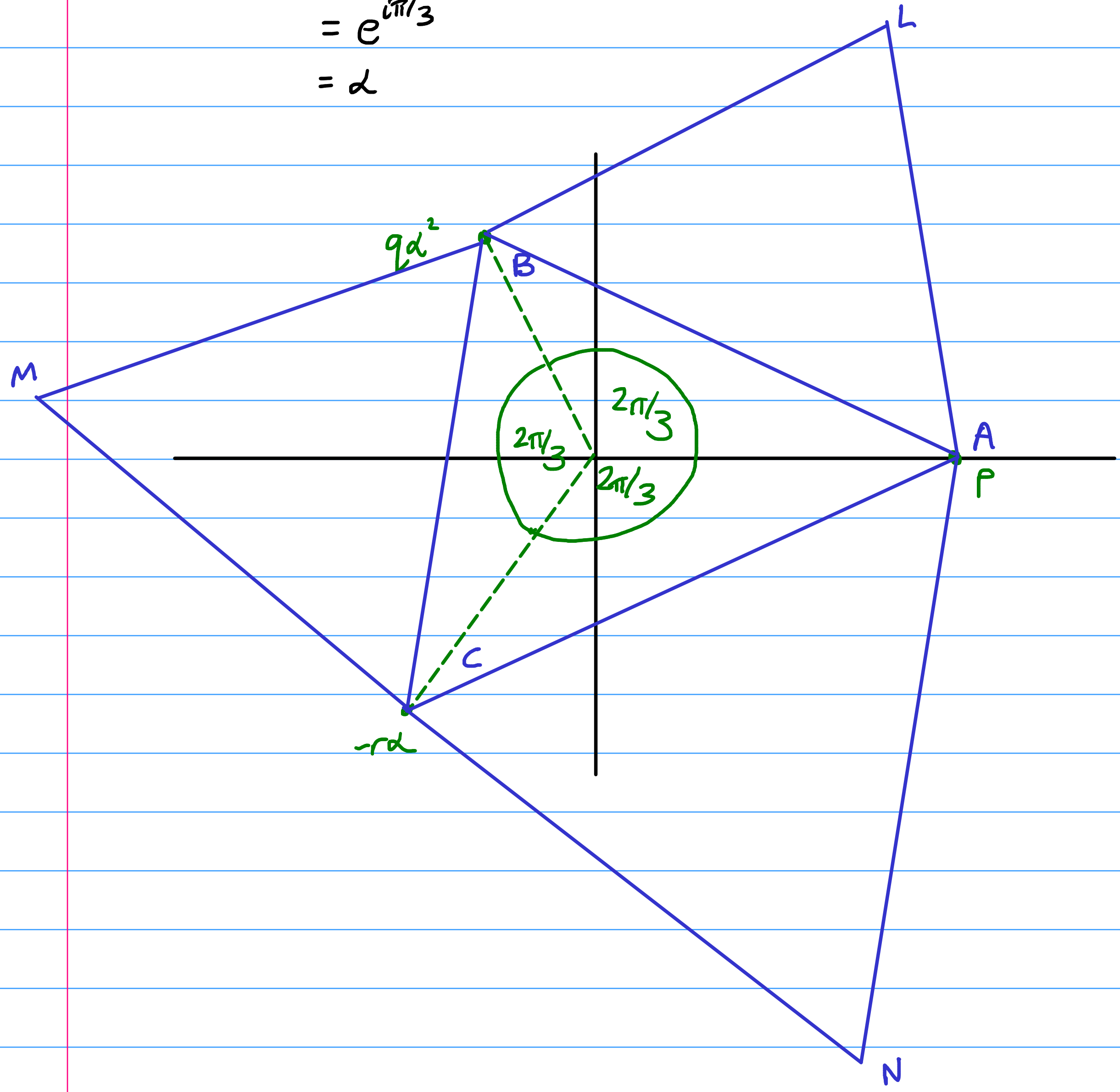
$= 1 - 1/2 + i\sin \pi/3$

$= \frac{1}{2} + i\sin\pi/3$

$= \cos\pi/3 + i\sin\pi/3$

$= e^{i\pi/3}$

$= \alpha$



To get N, we rotate C by $\pi/3$ about A. This is given by

$$(-r\alpha - p)e^{i\pi/3} + p$$

$= (-r\alpha - p)\alpha + p$
$= -r\alpha^2 + p(1-\alpha)$, as required.

Similarly, L is given by

$$(p - q\alpha^2)\cdot\alpha + q\alpha^2$$
$$= p\alpha + q\alpha^2(1-\alpha)$$
$$= p\alpha + q(\alpha^2+1) \qquad (\text{as } \alpha^3 = -1)$$

And M is given by
$$(q\alpha^2 + r\alpha)\cdot\alpha - r\alpha$$
$$= q\alpha^3 + r\alpha(\alpha - 1)$$
$$= -q + r\alpha(\alpha - 1)$$

Note further that $N = (1-\alpha)p - \alpha^2 r$
$= -\alpha^2 p - \alpha^2 r$
$= -\alpha^2(p+r)$

So $\overrightarrow{NB} = q\alpha^2 + \alpha^2(p+r)$
$= \alpha^2(p+q+r)$

Similarly $L = p\alpha + q(1+\alpha^2)$
$= \alpha(p+q)$
$\Rightarrow \overrightarrow{LC} = -\alpha(p+q+r)$

and $M = -q + r(\alpha^2 - \alpha)$
$= -(q+r)$
$\Rightarrow \overrightarrow{MA} = \alpha^2(p+q+r)$

As $|\alpha| = |\alpha^2| = 1$, these line segments all have length $p+q+r$. Further, $\arg B = \arg \alpha^2$ and $\arg N = -\arg(\alpha^2)$ so the line segment joining these points passes through the origin, and similar for the other pairs of points.

2000 STEP III Q4

$f(x) = x+c \Leftrightarrow (x+c)(x^2-1) = x(x-2)(x-a)$

$\Leftrightarrow \cancel{x^3} - x + cx^2 - c = \cancel{x^3} - 2x^2 - ax^2 + 2ax$

$\Leftrightarrow x^2(2+a+c) + x(-2a-1) - c = 0$

There is a solution to this equation provided that

$(2a+1)^2 - 4(2+a+c)(-c) \geqslant 0$

$\Leftrightarrow 4a^2+4a+1+8c+4ac+4c^2 \geqslant 0$

$\Leftrightarrow 4c^2 + c(8+4a) + (4a^2+4a+1) \geqslant 0$

For this to be true for all values of c, we require

$(8+4a)^2 - 4\times 4\times(4a^2+4a+1) \leqslant 0$

$\Leftrightarrow (2+a)^2 - (4a^2+4a+1) \leqslant 0$

$\Leftrightarrow 4+4a+a^2-4a^2-4a-1 \leqslant 0$

$\Leftrightarrow 3-3a^2 \leqslant 0$

$\Leftrightarrow a^2 \geqslant 1$

$\Leftrightarrow |a| \geqslant 1$

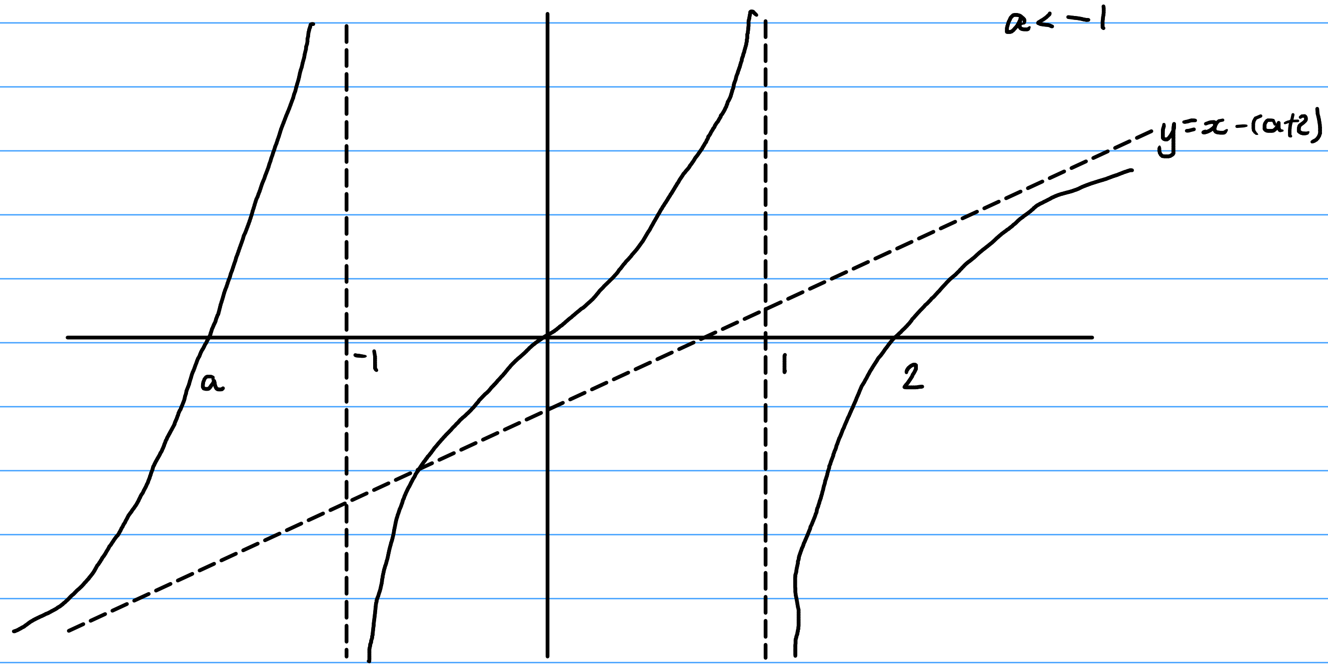
So if $|a| \geqslant 1$, there is a point of intersection for all c, but for $|a|<1$ there are values of c with no intersection.

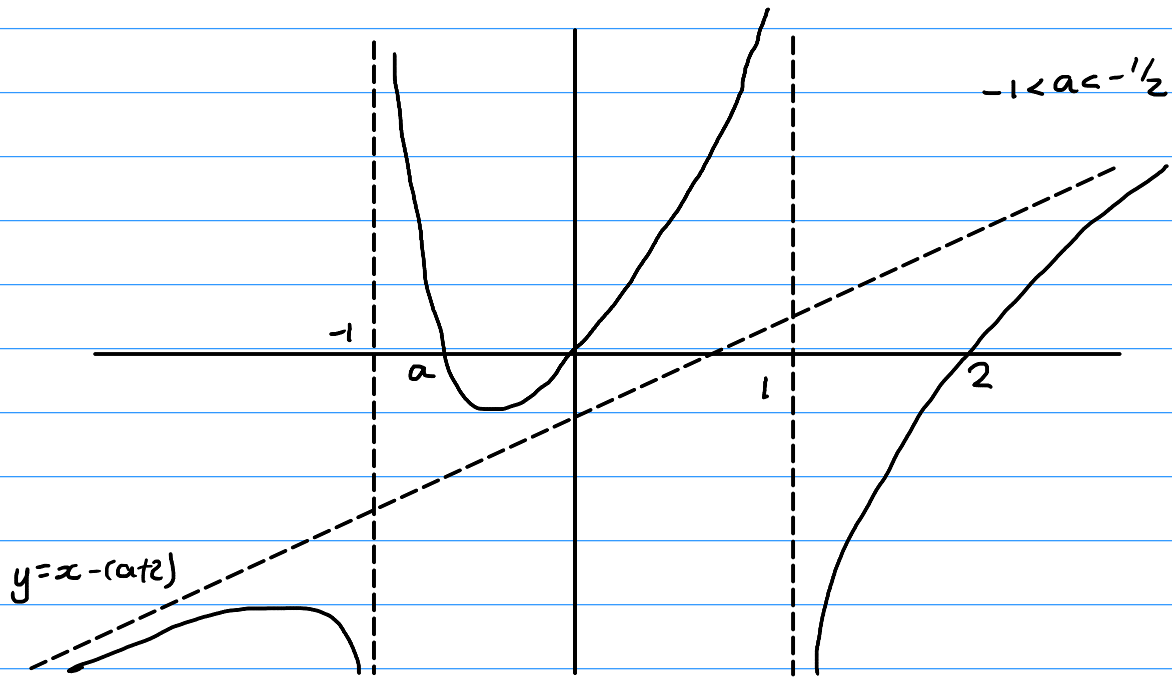
Now, $f(x) = \dfrac{x(x-2)(x-a)}{x^2-1}$

$$\begin{array}{r l}
 & x - (a+2) \\
x^2-1 \,\big) & x^3 - (a+2)x^2 + 2ax + 0 \\
 & \underline{x^3 + 0x^2 - x} \\
 & \quad -(a+2)x^2 + (2a+1)x + 0 \\
 & \quad \underline{-(a+2)x^2 + 0x + (a+2)} \\
 & \qquad (2a+1)x - (a+2)
\end{array}$$

So, $f(x) = x - (a+2) + \dfrac{(2a+1)x - (a+2)}{x^2 - 1}$

So the equation of the oblique asymptote is $y = x - (a+2)$





STEP III 2000 Q5

$\Delta(\underline{a},\underline{b}) = a_1 b_2 - a_2 b_1$

(i) Suppose $M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, so $M\underline{a} = \underline{p}$ and $M\underline{b} = \underline{q}$.

Then $a_1 w + a_2 x = p_1$ Because $\underline{a}$ and $\underline{b}$ are not parallel, the LHS of these
$b_1 w + b_2 x = q_1$ equations are not multiples of each other. Hence there is a (unique) solution for w and x.

Using the same argument, there is a unique solution for y and z. Hence M exists.

$$\Delta(\underline{a},\underline{b})\underline{c} + \Delta(\underline{c},\underline{a})\underline{b} + \Delta(\underline{b},\underline{c})\underline{a}$$

$$= \begin{pmatrix} a_1 b_2 c_1 - a_2 b_1 c_1 + c_1 a_2 b_1 - c_2 a_1 b_1 + b_1 c_2 a_1 - b_2 c_1 a_1 \\ a_1 b_2 c_2 - a_2 b_1 c_2 + c_1 a_2 b_2 - c_2 a_1 b_2 + b_1 c_2 a_2 - b_2 c_1 a_2 \end{pmatrix}$$

$$= \underline{0} \qquad (*)$$

We have $\Delta(\underline{p},\underline{q})$

$$= \Delta(M\underline{a}, M\underline{b})$$
$$= \Delta\left(\begin{pmatrix} wa_1 + xa_2 \\ ya_1 + za_2 \end{pmatrix}, \begin{pmatrix} wb_1 + xb_2 \\ yb_1 + zb_2 \end{pmatrix}\right)$$
$$= (wa_1 + xa_2)(yb_1 + zb_2) - (ya_1 + za_2)(wb_1 + xb_2)$$
$$= \cancel{wya_1b_1} + wza_1b_2 + xya_2b_1 + \cancel{xza_2b_2} - \cancel{wya_1b_1} - xya_1b_2 - wza_2b_1 - \cancel{xza_2b_2}$$
$$= (wz - xy)(a_1b_2 - a_2b_1)$$
$$= \det M \, \Delta(\underline{a},\underline{b})$$

Suppose that indeed $\underline{r} = M\underline{c}$. Then

$$\frac{\Delta(\underline{a},\underline{b})}{\Delta(\underline{p},\underline{q})} = \frac{\Delta(\underline{b},\underline{c})}{\Delta(\underline{q},\underline{r})} = \frac{\Delta(\underline{c},\underline{a})}{\Delta(\underline{r},\underline{p})} = \det M, \text{ as required.}$$

Otherwise, suppose the result holds.

We have $\Delta(\underline{a},\underline{b})\underline{c} + \Delta(\underline{c},\underline{a})\underline{b} + \Delta(\underline{b},\underline{c})\underline{a} = 0$

$$\Rightarrow \frac{\Delta(\underline{p},\underline{q})}{\Delta(\underline{a},\underline{b})} M\left(\Delta(\underline{a},\underline{b})\underline{c} + \Delta(\underline{c},\underline{a})\underline{b} + \Delta(\underline{b},\underline{c})\underline{a}\right) = 0$$

$$\Rightarrow \Delta(\underline{p},\underline{q}) M\underline{c} + \frac{\Delta(\underline{p},\underline{q})}{\Delta(\underline{a},\underline{b})}\Delta(\underline{c},\underline{a})\underline{q} + \frac{\Delta(\underline{p},\underline{q})}{\Delta(\underline{a},\underline{b})}\Delta(\underline{b},\underline{c})\underline{p} = 0$$

$$\Rightarrow \Delta(\underline{p},\underline{q}) M\underline{c} + \frac{\Delta(\underline{r},\underline{p})}{\Delta(\underline{c},\underline{a})}\Delta(\underline{c},\underline{a})\underline{q} + \frac{\Delta(\underline{q},\underline{r})}{\Delta(\underline{b},\underline{c})}\Delta(\underline{b},\underline{c})\underline{p} = 0 \qquad \text{(by the result)}$$

$$\Rightarrow M\underline{c} = \frac{-1}{\Delta(\underline{p},\underline{q})}\left(\Delta(\underline{q},\underline{r})\underline{p} + \Delta(\underline{r},\underline{p})\underline{q}\right)$$

$$= \underline{r} \qquad \text{(by } (*))$$

So $M\underline{c} = \underline{r}$, as required.

STEP III 2000 Q6

We have $x^4+px^2+qx+r=(x^2-ax+b)(x^2+ax+c)$

$\Rightarrow x^4+px^2+qx+r=x^4+x^2(c+b-a^2)+x(ab-ac)+bc$

So, $p=-a^2+b+c$
$q=ab-ac$
$r=bc$

Then, substituting $u=a$ into $u^3+2pu^2+(p^2-4r)u-q^2$ gives

$$a^6+2(-a^2+b+c)a^4+((-a^2+b+c)^2-4bc)a^2-(ab-ac)^2$$
$$=a^6-2a^6+2ba^4+2ca^4+(a^4+b^2+c^2-2a^2b-2a^2c+2bc-4bc)a^2-(a^2b^2-2a^2bc+a^2c^2)$$
$$=-a^6+a^4(2b+2c)+a^6+a^4(-2b-2c)+a^2(b^2+c^2-2bc)+a^2(2bc-a^2-b^2)$$
$$=0$$

$a^2\geq 0$, so this equation always has a non-negative root.
If $p=-1, q=-6, r=15$, then $u^3-2u^2-59u-36=0$
Substituting $u=9$ gives $9^3-2\times 9^2-59\times 9-4\times 9$
$=9(81-18-59-4)$
$=0$, as required.

Now consider $y^4-8y^3+23y^2-34y+39$

Note that $(y-2)^4=y^4-8y^3+24y^2-32y+16$, so the expression becomes

$$(y-2)^4-y^2-2y+23$$

Substituting $x=y-2$, we have

$$x^4 - (x+2)^2 - 2(x+2) + 23$$
$$= x^4 - x^2 - 4x - 4 - 2x - 4 + 23$$
$$= x^4 - x^2 - 6x + 15$$

This is an expression of the earlier form with $p=-1$, $q=-6$, $r=15$. We have seen that $a^2=9$ in this case. Take $a=3$ ($a=-3$ just swops b and c).

Then $-6 = 3(b-c)$ and $bc=15$
$\Rightarrow b-c=-2$ $\Rightarrow b=3$ $c=5$

So our quartic factors to
$$(x^2-3x+3)(x^2+3x+5)$$
$$= ((y-2)^2 - 3(y-2)+3)((y-2)^2+3(y-2)+5)$$
$$= (y^2-4y+4-3y+6+3)(y^2-4y+4+3y-6+5)$$
$$= (y^2-7y+13)(y^2-y+3)$$

STEP III 2000 Q7

$$\left(1+\frac{1}{n}\right)^n = 1 + n\times\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \dots + \frac{1}{n^n}$$
$$< 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \text{—} + \frac{1}{n!}$$
$$< e$$

$$P(n) = \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{9}{8}\cdot \dots \cdot\frac{2^n+1}{2^n}$$

$$\le \left(\frac{3/2 + 5/4 + 9/8 + \text{—} + 2^n+1/2^n}{n}\right)^n \qquad \text{(AM-GM inequality)}$$

$$= \left(\frac{3\times2^{n-1} + 5\times2^{n-2} + 9\times2^{n-3} + \text{—} + 2^n+1}{n\times2^n}\right)^n$$

$$= \left(\frac{(2\times2^{n-1}+2^{n-1}) + (2^2\times2^{n-2}+2^{n-2}) + (2^3\times2^{n-3}+2^{n-3}) + \text{—} + 2^n+1}{n\times2^n}\right)^n$$

$$= \left(\frac{n\times2^n + (2^0+2^1+\text{—}+2^{n-1})}{n\times2^n}\right)^n$$

$$= \left(1+\frac{2^n-1}{n\times2^n}\right)^n$$

$$< \left(1+\frac{2^n}{n\times2^n}\right)^n$$
$$= (1+1/n)^n$$
$$< e$$

$P(n)$ is clearly increasing in n, and $P(n)<e$ for all n, so $P(n)$ reaches a limit as $n\to\infty$.

$P(3) = \frac{3}{2}\times\frac{5}{4}\times\frac{9}{8} = \frac{135}{64} > 2.$

So $\quad 2 < L \le e.$

STEP III 2000 Q8

$$a_n = 1 + \frac{a_{n-1}^2}{a_{n-2}}, \quad a_0 = 1, \quad a_1 = 1$$

Claim: $a_n = 3a_{n-1} - a_{n-2}$

$n=2$: $a_2 = 1 + \frac{1^2}{1} = 2$ or $a_2 = 3\times 1 - 1 = 2$ ✓

$n=3$: $a_3 = \frac{1+2^2}{1} = 5$ or $a_3 = 3\times 2 - 1 = 5$ ✓

Suppose true for $n=k-1$ and $n=k-2$.

Then
$$
\begin{aligned}
a_k &= \frac{1+a_{k-1}^2}{a_{k-2}} \\
&= \frac{1+(3a_{k-2}-a_{k-3})^2}{a_{k-2}} \\
&= \frac{1+9a_{k-2}^2 - 6a_{k-2}a_{k-3} + a_{k-3}^2}{a_{k-2}} \\
&= \frac{1+a_{k-3}^2}{a_{k-2}} + 9a_{k-2} - 6a_{k-3} \\
&= \frac{a_{k-4}}{a_{k-4}} \cdot \frac{1+a_{k-3}^2}{a_{k-2}} + 9a_{k-2} - 6a_{k-3} \\
&= \frac{a_{k-4}}{a_{k-2}} \times a_{k-2} + 9a_{k-2} - 6a_{k-3} \\
&= a_{k-4} + 9a_{k-2} - 6a_{k-3} \\
&= (3a_{k-3} - a_{k-2}) + 9a_{k-2} - 6a_{k-3} \\
&= 8a_{k-2} - 3a_{k-3} \\
&= 8a_{k-2} - 3(3a_{k-2} - a_{k-1}) \\
&= 3a_{k-1} - a_{k-2}
\end{aligned}
$$

True for $n=2$, $n=3$, and if true for $n=k-1$, $n=k-2$, then true for $n=k$. So true for all $n \geqslant 2$.

Suppose $a_k = \mu \times \lambda^k$

Then $\mu \lambda^2 \lambda^{k-2} = 3\mu\lambda\lambda^{k-2} - \mu\lambda^{k-2}$

$\Rightarrow \lambda^2 - 3\lambda + 1 = 0$

$\Rightarrow \lambda = \frac{3 \pm \sqrt{9-4}}{2}$

$\Rightarrow \lambda = \frac{3 \pm \sqrt{5}}{2}$

However, $\left(\frac{1+\sqrt{5}}{2}\right)^{2n-1}$

$= c \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^2\right)^n$

$= c\left(\frac{1+5+2\sqrt{5}}{4}\right)$

$= c\left(\frac{3+\sqrt{5}}{2}\right)^n$

where c is a constant.

$\left(\frac{1-\sqrt{5}}{2}\right)^{-(2n-1)}$

$= c \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^{-2}\right)^n$

$= c \cdot \left(\frac{2}{3+\sqrt{5}}\right)^n$

$= c\left(\frac{2(3-\sqrt{5})}{9-5}\right)^n$

$= c\left(\frac{6-2\sqrt{5}}{4}\right)^n$

$= c\left(\frac{3-\sqrt{5}}{2}\right)^n$

So, equivalently, $a_k = A\alpha^{2n-1} + B\alpha^{-(2n-1)}$ where $\alpha = \frac{1+\sqrt{5}}{2}$

For $n=1$, $1 = A\frac{1+\sqrt{5}}{2} + B\frac{-1+\sqrt{5}}{2}$

$n=2$, $2 = A\alpha^3 + B\alpha^{-3}$

$= A(2+\sqrt{5}) + B(-2+\sqrt{5})$

Hence $A = \frac{2+B(2-\sqrt{5})}{2+\sqrt{5}}$

$= \frac{2(\sqrt{5}-2) - B(\sqrt{5}-2)^2}{1}$

$= 2\sqrt{5} - 4 + (4\sqrt{5} - 9)B$

Substituting this into the first equation

$$2 = (1+\sqrt{5})(2\sqrt{5}-4) + (1+\sqrt{5})(4\sqrt{5}-9)B + B(-1+\sqrt{5})$$

$$\Rightarrow 2 = 2\sqrt{5} - 4\sqrt{5} - 4 + 10 + B(4\sqrt{5} - 9\sqrt{5} - 9 + 20 - 1 + \sqrt{5})$$

$$\Rightarrow -4 + 2\sqrt{5} = B(-4\sqrt{5} + 10)$$

$$= \sqrt{5}B(-4 + 2\sqrt{5})$$

$$\Rightarrow \sqrt{5}B = 1$$

$$\Rightarrow B = 1/\sqrt{5}$$

$$A = 2\sqrt{5} - 4 + (4\sqrt{5} - 9) \cdot \frac{1}{\sqrt{5}}$$

$$= 2\sqrt{5} - 4 + 4 - 9/\sqrt{5}$$

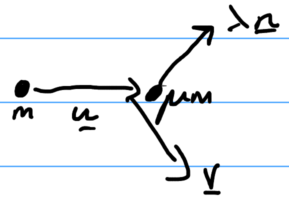
$$= 10/\sqrt{5} - 9/\sqrt{5}$$

$$= \sqrt{5}/5$$

$$= 1/\sqrt{5}$$

So, $a_n = \dfrac{\alpha^{2n-1} + \alpha^{-(2n-1)}}{\sqrt{5}}$

STEP III 2000 Q9



Conservation of Momentum: $m\underline{u} = \mu m \lambda \underline{n} + m\underline{v}$

$$\Rightarrow \underline{u} = \mu\lambda\underline{n} + \underline{v}$$

Restitution: $\underline{u} = \lambda\underline{n} - \underline{v}$

Dotting both equations with $\underline{n}$ gives

$$\underline{u}\cdot\underline{n} = \mu\lambda + \underline{v}\cdot\underline{n} \qquad (\text{noting } \underline{n}\cdot\underline{n} = 1)$$
$$\underline{u}\cdot\underline{n} = \lambda - \underline{v}\cdot\underline{n}$$

$$\Rightarrow 2\underline{u}\cdot\underline{n} = (1+\mu)\lambda$$

$$\Rightarrow \lambda = \frac{2\underline{u}\cdot\underline{n}}{1+\mu}$$

Energy: $\frac{1}{2}m\underline{u}\cdot\underline{u} = \frac{1}{2}\mu m(\lambda\underline{n}\cdot\lambda\underline{n}) + \frac{1}{2}m\underline{v}\cdot\underline{v}$

$= \frac{1}{2}\mu m\lambda^2 + \frac{1}{2}\mu m\lambda^2$ as both have equal KE

$= \mu m\lambda^2$

$$\Rightarrow \frac{1}{2}m|\underline{u}|^2 = \mu m\left(\frac{2|\underline{u}||\underline{n}|\cos\theta}{1+\mu}\right)^2$$

$$\Rightarrow \frac{1}{8} = \frac{\mu\cos^2\theta}{(1+\mu)^2}$$

$$\Rightarrow \cos\theta = \pm\frac{1+\mu}{\sqrt{8\mu}}$$

But $-1 \leq \cos\theta \leq 1$, so

$$-1 \leq \frac{1+\mu}{\sqrt{8\mu}} \leq 1$$

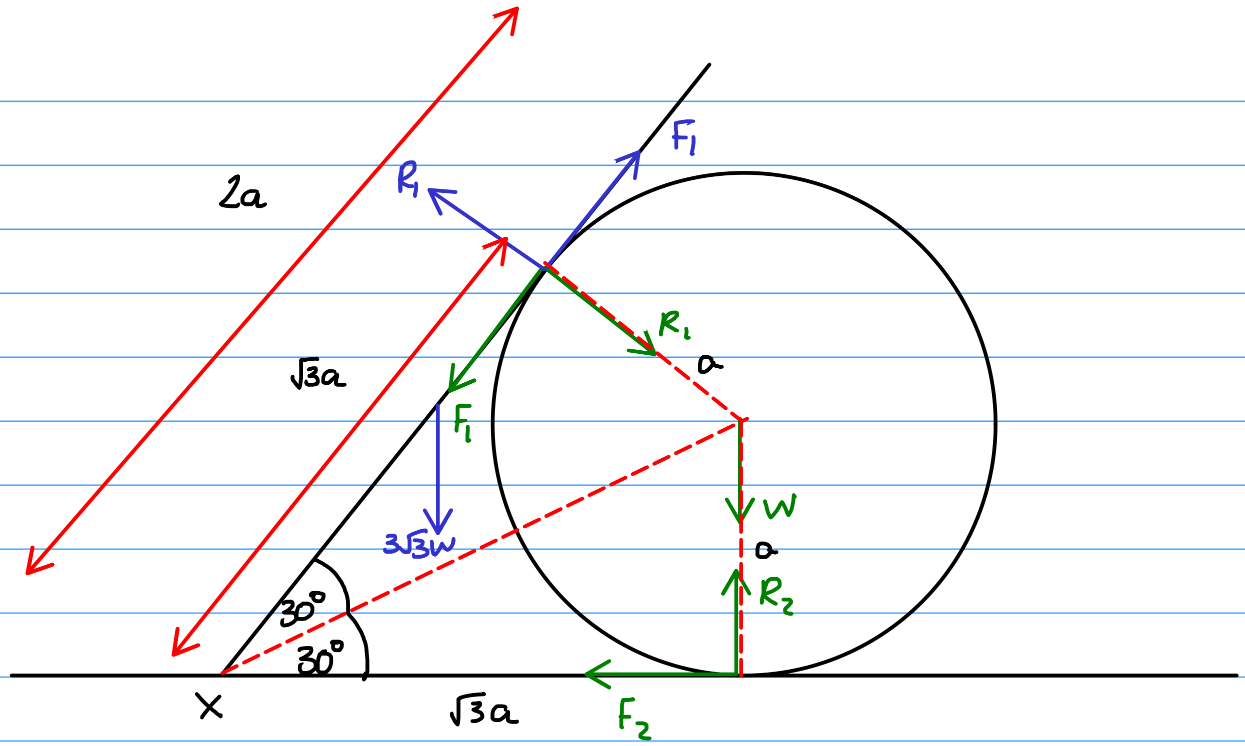
$$\Rightarrow \quad -\sqrt{8\mu} \leq 1+\mu \leq \sqrt{8\mu}$$

$$\Rightarrow \quad (1+\mu)^2 \leq 8\mu$$

$$\Rightarrow \quad \mu^2 - 6\mu + 1 \leq 0$$

$$\Rightarrow \quad (\mu-3)^2 \leq 8$$

$$\Rightarrow \quad 3-\sqrt{8} \leq \mu \leq 3+\sqrt{8}$$



Forces acting on the sphere are green, and forces acting on the beam are blue.

Taking moments on the sphere about its centre gives $F_1 = F_2 = F$

Taking moments on the beam about X gives $3\cancel{\sqrt{3}}W \cdot \cancel{a}\cos(60°) = R_1 \cdot \cancel{\sqrt{3}a}$

$$\Rightarrow \frac{3W}{2} = R_1$$

Resolving forces vertically for the sphere,

$$F_1 \sin(60°) + R_1 \cos(60°) + W = R_2$$

$$\Rightarrow \frac{\sqrt{3}}{2}F + \frac{3W}{2} \cdot \frac{1}{2} + W = R_2$$

$$\Rightarrow R_2 = \frac{7}{4}W + \frac{\sqrt{3}}{2}F$$

Resolving forces horizontally for the sphere,

$$F_2 + F_1 \cos(60°) = R_1 \sin(60°)$$

$$\Rightarrow F + F \cdot \frac{1}{2} = \frac{3W}{2} \cdot \frac{\sqrt{3}}{2}$$

$$\Rightarrow F = \frac{\sqrt{3}}{2}W$$

So $R_2 = \frac{7}{4}W + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}W$
$= \frac{7}{4}W + \frac{3}{4}W$
$= \frac{5}{2}W$

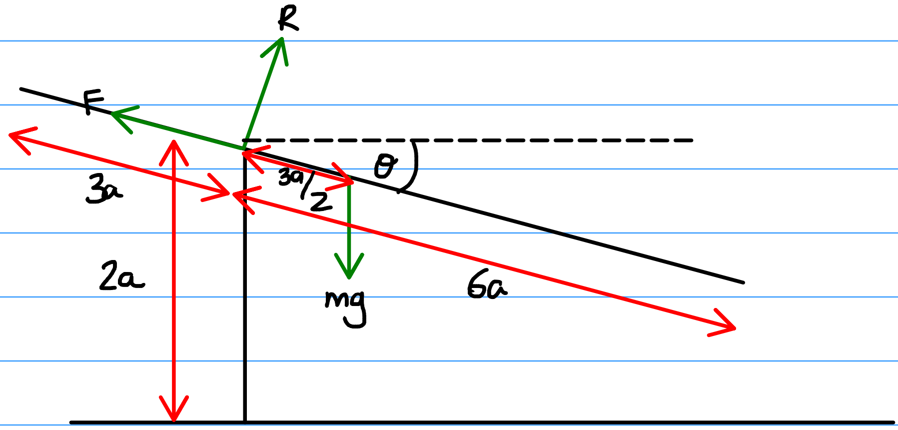
The sphere is on the point of slipping at both points, so

$$\mu_1 = \frac{F}{R_1} = \frac{\sqrt{3}/2\,W}{3/2\,W} = \frac{\sqrt{3}}{3}$$

$$\mu_2 = \frac{F}{R_2} = \frac{\sqrt{3}/2\,W}{5/2\,W} = \frac{\sqrt{3}}{5}$$

STEP III 2000 Q11

When the rod has rotated by θ,



About the pivot, the moment of inertia of the rod is

$$\frac{1}{3}\times\frac{2}{3}m\times(6a)^2+\frac{1}{3}\times\frac{1}{3}m\times(3a)^2$$

$$= 9ma^2 \qquad \text{(using MoI of rod length } L \text{ mass } m \text{ as } \tfrac{1}{3}ML^2\text{)}$$

By conservation of energy,

$$\frac{1}{2}\times 9ma^2\times\dot{\theta}^2=\frac{3a}{2}mg\sin\theta$$

KE gain = GPE loss

$$\Rightarrow 3a\dot{\theta}^2=g\sin\theta \qquad (\dagger)$$

Differentiating, $6a\dot{\theta}\ddot{\theta}=g\cos\theta\,\dot{\theta}$

$$\Rightarrow 6a\ddot{\theta}=g\cos\theta \qquad (*)$$

Resolving perpendicular to the beam,

$$mg\cos\theta-R=\frac{3}{2}ma\ddot{\theta}$$

$$=\frac{mg}{4}\cos\theta \quad \text{by } (*)$$

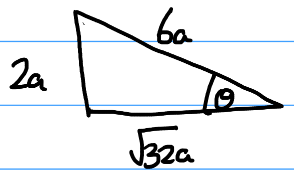
Hence $R = \frac{3mg\cos\theta}{4}$

Resolving parallel to the beam,

$$F - mg\sin\theta = mr\omega^2$$
$$= \frac{3}{2}ma\dot{\theta}^2$$
$$= \frac{1}{2}mg\sin\theta \qquad \text{(by †)}$$
$$\Rightarrow F = \frac{3}{2}mg\sin\theta$$

For no slipping $\mu \geq \frac{F}{R} = \frac{3mg\sin\theta}{2} \times \frac{4}{3mg\cos\theta}$
$$= 2\tan\theta$$

This is increasing in θ, and at a maximum as the end of the rod hits the floor.
At this point,



$\Rightarrow \tan\theta = \frac{2}{\sqrt{32}} = \frac{1}{2\sqrt{2}}$

So $\mu \geq 2 \times \frac{1}{2\sqrt{2}}$
$= \sqrt{2}/2$

STEP III 2000 Q12

$$P(\leq 2 \text{ winners} + \text{me})$$
$$= \binom{2N-1}{2}\left(\frac{1}{N}\right)^2\left(\frac{N-1}{N}\right)^{2N-3} + \binom{2N-1}{1}\left(\frac{1}{N}\right)\left(\frac{N-1}{N}\right)^{2N-2} + \binom{2N-1}{0}\left(1-\frac{1}{N}\right)^{2N-1}$$

$$= \left(1-\frac{1}{N}\right)^{2N-3}\left[\frac{(2N-1)(2N-2)}{2N^2} + \frac{(2N-1)}{N}\left(1-\frac{1}{N}\right) + \left(1-\frac{1}{N}\right)^2\right]$$

$$= \left(1-\frac{1}{N}\right)^{2N-3}\left[2-\frac{3}{N}+\frac{1}{N^2}+2-\frac{3}{N}+\frac{1}{N^2}+1-\frac{2}{N}+\frac{1}{N^2}\right]$$

$$= \left(1-\frac{1}{N}\right)^{2N-3}\left(5-\frac{8}{N}+\frac{3}{N^2}\right)$$

$$= \left[\left(1-\frac{1}{N}\right)^N\right]^2 \cdot \frac{5-\frac{8}{N}+\frac{3}{N^2}}{(1-1/N)^3}$$

As $N \to \infty$, this $\to e^{-2} \times \frac{5}{1}$
$$= 5e^{-2}$$

$e \approx 2.8 = \frac{14}{5}$ so $5e^{-2} \approx 5 \times \frac{25}{196}$
$$= \frac{125}{196}$$
$$= \frac{375}{3\times 196}$$
$$= \frac{392}{3\times196} - \frac{17}{3\times196}$$
$$= \frac{2}{3} - \frac{17}{588}$$
$$\approx \frac{2}{3}$$

If numbers were chosen by players, some numbers are more likely to be chosen by others. So knowing I hold a winning ticket means it's more likely other players chose this number. So this probability will decrease, as there is a higher probability of more winners.

Each other player is a winner independently with probability $p = 1/N$, so the total number of winners $\sim 1 + B(2N-1, 1/N)$, with mean

$$
\begin{aligned}
& 1 + (2N-1)(1/N) \\
= \; & 1 + 2 - 1/N \\
= \; & 3 - 1/N
\end{aligned}
$$

STEP III 2000 Q13

P(first 6 on r^{th} roll) = P(no sixes in $(r-1)$ rolls)P(≥ 1 6 on r^{th} roll)

$$= q^{n(r-1)} \times (1-q^n)$$
$$= q^{nr-n} q^n (q^{-n}-1)$$
$$= q^{nr}(q^{-n}-1)$$

The PGF is $E(t^X) = \sum_{r=1}^{\infty} q^{nr}(q^{-n}-1)t^r$

$$= (q^{-n}-1)\sum_{r=1}^{\infty}(tq^n)^r$$
$$= (q^{-n}-1)\cdot \frac{tq^n}{1-tq^n}$$
$$= \frac{(1-q^n)t}{1-q^nt}$$

Now, $G(t) = E(t^X) = \sum P(R=r)t^r$

$$\frac{dG}{dt} = \sum rt^{r-1}P(R=r)$$

So $G'(1) = \sum rP(R=r)$

$$= E(R)$$

$$G'(t) = \frac{(1-q^n)(1-q^nt)+q^n(1-q^n)t}{(1-q^nt)^2}$$

So $G'(1) = \dfrac{(1-q^n)^2+q^n(1-q^n)}{(1-q^n)^2}$

$$= \frac{1-q^n+q^n}{1-q^n}$$

$$= \frac{1}{1-q^n} = ER$$

$$n=2,\ p=1/6 = \frac{1}{1-(5/6)^2} = \frac{1}{1-25/36}$$
$$= \frac{36}{11}$$

P(last dice shows on r^{th} roll) = P(last shows $\le r$) − P(last shows $\le r-1$)

= P(each dice shows 6 $\le r^{th}$ roll) − P(each shows $\le (r-1)^{th}$ roll)

$$= (1-q^r)^n - (1-q^{r-1})^n$$

For $n=2$, this is $(1-q^r)^2 - (1-q^{r-1})^2$

$$= 1-2q^r+q^{2r}-1+2q^{r-1}-q^{2r-2}$$
$$= q^{2r}-q^{2r-2}-2q^r+2q^{r-1}$$

So $G(t)=E(t^R)=\sum_{r=1}^{\infty}\left(q^{2r}-q^{2r-2}-2q^r+2q^{r-1}\right)t^r$

$$=\sum_{r=1}^{\infty}(tq^2)^r-\frac{1}{q^2}(tq^2)^r-2(tq)^r+\frac{2}{q}(tq)^{r-1}$$
$$=\left(1-\frac{1}{q^2}\right)\sum_{r=1}^{\infty}(tq^2)^r-2\left(1-\frac{1}{q}\right)\sum_{r=1}^{\infty}(tq)^r$$
$$=\left(1-\frac{1}{q^2}\right)\frac{tq^2}{1-tq^2}-2\left(1-\frac{1}{q}\right)\frac{tq}{1-tq}$$

So, $G'(t)=\left(1-\frac{1}{q^2}\right)\frac{q^2(1-tq^2)+tq^4}{(1-tq^2)^2}-2\left(1-\frac{1}{q}\right)\frac{q(1-tq)+tq^2}{(1-tq)^2}$

So $G'(1)=\left(1-\frac{1}{q^2}\right)\frac{q^2-q^4+q^4}{(1-q^2)^2}-2\left(1-\frac{1}{q}\right)\frac{q-q^2+q^2}{(1-q)^2}$

$$= \left(1-\frac{1}{q^2}\right)\frac{q^2}{(1-q^2)^2} - 2\left(1-\frac{1}{q}\right)\frac{q}{(1-q)^2}$$

$$= \frac{q^2-1}{(1-q^2)^2} - \frac{2(q-1)}{(1-q)^2}$$

$$= \frac{-1}{1-q^2} + \frac{2}{1-q}$$

$$= \frac{-1+2(1+q)}{1-q^2}$$

$$= \frac{1+2q}{1-q^2} = ES$$

If $q = 5/6$, $ES = \dfrac{1+5/3}{1-25/36}$

$$= \frac{36+5\times 12}{36-25}$$

$$= \frac{96}{11}$$

STEP III 2000 Q14

$X \backslash Y$	y_1	y_2
x_1	a	$q-a$
x_2	$p-a$	$1+a-p-q$

If $E(XY) = EXEY$ then

$$ax_1y_1 + (q-a)x_1y_2 + (p-a)x_2y_1 + (1+a-p-q)x_2y_2$$

$$= (qx_1 + (1-q)x_2)(py_1 + (1-p)y_2)$$
$$= pqx_1y_1 + q(1-p)x_1y_2 + p(1-q)x_2y_1 + (1-p)(1-q)x_2y_2$$

$$\Rightarrow (a-pq)x_1y_1 + (pq-a)x_1y_2 + (pq-a)x_2y_1 + (a-pq)x_2y_2 = 0$$

$$\Rightarrow (a-pq)(x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2) = 0$$
$$\Rightarrow (a-pq)(x_1-x_2)(y_1-y_2) = 0$$

Because $x_1 \neq x_2$ and $y_1 \neq y_2$, we must have $a = pq$.

So $P(X=x_1 | Y=y_1) = \dfrac{P(X=x_1, Y=y_1)}{P(Y=y_1)} = \dfrac{a}{p} = \dfrac{pq}{p} = q$

$P(X=x_1 | Y=y_2) = \dfrac{q-a}{1-p} = \dfrac{q-pq}{1-p} = q$

So X and Y are independent.

Let A take values in $\{-1, 0, 1\}$ with equal probability.
If $A=0$, then $B=0$
If $A \neq 0$, then $B = A$ or $-A$, each with probability $1/2$.

Then clearly A & B are not idependent, as $P(B=0 | A=0) = 1 \neq P(B=0) = 1/3$

But clearly $EA = EB = 0$, and

$$EAB = \frac{1}{3} \times 0 \times 0 + \frac{1}{6} \times -1 \times -1 + \frac{1}{6} \times 1 \times -1 + \frac{1}{6} \times -1 \times 1 + \frac{1}{6} \times 1 \times 1$$
$$= 0.$$