STEP II 2000 Q1

we claim that $\frac{1}{N} \equiv \frac{1}{N+1} + \frac{1}{N(N+1)}$

We have $\frac{1}{N+1} + \frac{1}{N(N+1)} = \frac{N}{N(N+1)} + \frac{1}{N(N+1)}$

$$= \frac{(N+1)}{N(N+1)}$$

$$= \frac{1}{N}, \text{ as required.}$$

So for any unit fraction $\frac{1}{N}$, we have $\frac{1}{N} = \frac{1}{N+1} + \frac{1}{N(N+1)}$, the sum of two distinct unit fractions (provided $N > 1$)

Now $\frac{1}{N} = \frac{1}{a} + \frac{1}{b}$

$$\Rightarrow \frac{1}{N} = \frac{a+b}{ab}$$

$$\Rightarrow N = \frac{ab}{a+b}$$

$$\Rightarrow aN + bN = ab$$

$$\Rightarrow N^2 = N^2 - an - bn + ab$$

$$= (a-N)(b-N)$$

If N is prime, we must have $(a-N) = (b-N) = N$ ✗ (a, b not distinct)

or $a - N = 1$, $b - N = N^2$ (wlog $a < b$)

So, $a = N+1$ and $b = N(N+1)$ is the only way of expressing $\frac{1}{N}$ in the form $\frac{1}{a} + \frac{1}{b}$ with $a \neq b$.

Now consider $\frac{2}{N} = \frac{1}{a} + \frac{1}{b}$ with $a < b$, N prime, $N > 2$.

So $N = \frac{2ab}{a+b}$ (similar to before)

$\Rightarrow aN + bN = 2ab$

$\Rightarrow 0 = 4ab - 2aN - 2bN$

$\Rightarrow N^2 = 4ab - 2aN - 2bN + N^2$

$= (2a - N)(2b - N)$

As before, we have $2a - N = 1$ $\quad 2b - N = N^2$

$\Rightarrow a = \frac{N+1}{2}$ $\quad \Rightarrow b = \frac{N(N+1)}{2}$

N is odd (as prime), so $a, b \in \mathbb{Z}$. Then, by the same argument as previously, $\frac{2}{N} = \frac{1}{a} + \frac{1}{b}$, and this is the unique solution.

STEP II 2000 Q2

Suppose $p(x) = (x-a)^2 q(x)$ where $q(x)$ is a polynomial.

Then $p'(x) = 2(x-a)q(x) + (x-a)^2 q'(x)$

So $p'(a) = 0$.

Suppose $p(x) = (x-a)^4 q(x)$ where $q(x)$ is a polynomial.

Then $p'(x) = 4(x-a)^3 q(x) + (x-a)^4 q'(x)$

$p''(x) = 12(x-a)^2 q(x) + 4(x-a)^3 q'(x) + 4(x-a)^3 q'(x) + (x-a)^4 q''(x)$

$= 12(x-a)^2 q(x) + 8(x-a)^3 q'(x) + (x-a)^4 q''(x)$

$p'''(x) = 24(x-a)q(x) + 36(x-a)^2 q'(x) + 12(x-a)^3 q''(x) + (x-a)^4 q'''(x)$

So $p'''(a) = 0$

Now $p(x) = x^6 + 4x^5 - 5x^4 - 40x^3 - 40x^2 + 32x + k$

$p'(x) = 6x^5 + 20x^4 - 20x^3 - 120x^2 - 80x + 32$

$p''(x) = 30x^4 + 80x^3 - 60x^2 - 240x - 80$

$p'''(x) = 120x^3 + 240x^2 - 120x - 240 = 0$

$p'''(x) = 0 \Rightarrow x^3 + 2x^2 - x - 2 = 0$

By inspection, the solutions are $x = 1$, $x = -1$, $x = -2$.

Note, we must also have $p'(a) = p''(a) = 0$

$p'(1) = 6 + 20 - 20 - 120 - 80 + 32 = -162 \neq 0$

$p'(-1) = -6 + 20 + 20 - 120 + 80 + 32 = 26 \neq 0$

$p'(-2) = -6 \times 32 + 10 \times 32 + 5 \times 32 - 15 \times 32 + 5 \times 32 + 32 = 0$

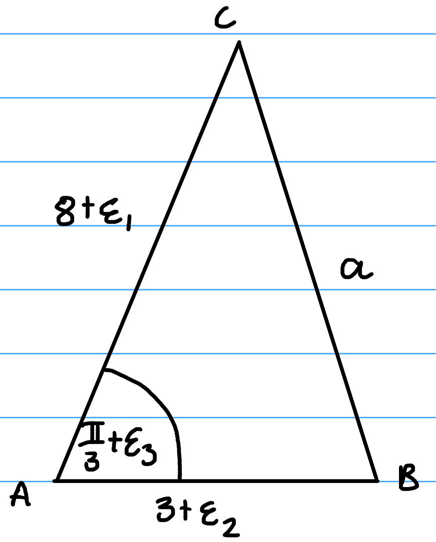
$p''(-2) = 10(48 - 64 - 24 + 48 - 8) = 0$

$p(-2)=0 \Rightarrow 64-128-80+320-160-64+k=0.$

$\Rightarrow k=48$

Since we know $p(x)=(x-a)^4q(x)$, and the only possible value of a is -2, we have $p(x)=(x+2)^4q(x)$ and $k=48$.

STEP II 2000 Q3



We have $a^2=b^2+c^2-2bc\cos A$

$$=(8+\varepsilon_1)^2+(3+\varepsilon_2)^2-2(8+\varepsilon_1)(3+\varepsilon_2)\cos(\pi/3+\varepsilon_3)$$

$$\approx 64+16\varepsilon_1+9+6\varepsilon_2-2(24+8\varepsilon_2+3\varepsilon_1)(\cos\tfrac{\pi}{3}\cos\varepsilon_3-\sin\tfrac{\pi}{3}\sin\varepsilon_3)$$

(ignoring quadratic and higher terms in ε)

$$\approx 73+16\varepsilon_1+6\varepsilon_2-(24+8\varepsilon_2+3\varepsilon_1)(1\times 1-\sqrt{3}\varepsilon_3)$$

$$\approx 73+16\varepsilon_1+6\varepsilon_2-24+24\sqrt{3}\varepsilon_3-8\varepsilon_2-3\varepsilon_1$$

$$=49+13\varepsilon_1-2\varepsilon_2+24\sqrt{3}\varepsilon_3$$

$$=49+\lambda,\quad \text{where } \lambda=13\varepsilon_1-2\varepsilon_2+24\sqrt{3}\varepsilon_3$$

So $a=(49+\lambda)^{1/2}$

$$=7\left(1+\frac{\lambda}{49}\right)^{1/2}$$

$$\approx 7\left(1+\frac{1}{2}\times\frac{\lambda}{49}\right)$$ (ignoring uadratic and higher terms in λ)

$$=7+\frac{\lambda}{14}$$

$=7+\eta$, as required.

Note that η is symmetric about 0 (because the ε are), so $\max(\eta)=-\min(\eta)$.

$$\text{Max } \eta=\frac{2.6\times10^{-2}+9.8\times10^{-2}+7.2\times10^{-2}}{14}$$

$$= \frac{19.6 \times 10^{-2}}{14}$$

$$= 1.4 \times 10^{-2}$$

So $-1.4 \times 10^{-2} \leq \eta \leq 1.4 \times 10^{-2}$

STEP II 2000 Q4

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$
$$= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi$$
$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$

<u>Claim</u>

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof via induction

The $n=1$ case is clear.

Suppose true for $n=k$.

Then for $n=k+1$,

$$LHS = (\cos\theta + i\sin\theta)^{k+1}$$
$$= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k$$
$$= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta) \quad \text{by assumption}$$
$$= \cos(\theta + k\theta) + i\sin(\theta + k\theta) \quad \text{by earlier result}$$
$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

So the result is true by induction.

Now, $(5-i)^2(1+i)$
$$= (24-10i)(1+i)$$
$$= 34 + 14i$$

So $\arg[(5-i)^2(1+i)] = \arg(34+14i)$

$\Rightarrow 2\arg(5-i) + \arg(1+i) = \arg(34+14i)$

$2\arctan\left(\frac{-1}{5}\right) + \arctan 1 = \arctan\frac{7}{17}$

$-2\arctan\left(\frac{1}{5}\right) + \pi/4 = \arctan\frac{7}{17}$

$\Rightarrow \arctan\frac{7}{17} + 2\arctan\frac{1}{5} = \pi/4$

Consider $(4-i)^3(20-i)(1+i)$

$$= (64-48i-12+i)(21+19i)$$

$$= (51-47i)(21+19i)$$

$$= 1985+i$$

$$\Rightarrow 3\arg(4-i)+\arg(20-i)+\arg(1+i) = \arg(1985+i)$$

$$\Rightarrow -3\arctan\tfrac{1}{4} - \arctan\tfrac{1}{20} + \pi/4 = \arctan\tfrac{1}{1985}$$

$$\Rightarrow \pi/4 = 3\arctan\tfrac{1}{4} + \arctan\tfrac{1}{20} + \arctan\tfrac{1}{1985}$$

STEP II 2000 Q5

$$g(\lambda) = \int_0^1 (f(x) - \lambda x)^2 dx = \int_0^1 f(x)^2 dx - 2\lambda \int_0^1 x f(x) dx + \lambda^2 \int_0^1 x^2 dx$$

$$= \int_0^1 f(x)^2 dx - 2\lambda \int_0^1 x f(x) dx + \frac{1}{3}\lambda^2$$

$$\frac{dg}{d\lambda} = 0 - 2\int_0^1 x f(x) dx + \frac{2}{3}\lambda = 0$$

$$\Longrightarrow \lambda = 3\int_0^1 x f(x) dx$$

$$R^2 = \int_0^1 (f(x))^2 - 2\lambda \int_0^1 x f(x) dx + \frac{1}{3}\lambda^2$$

$$= \int_0^1 (f(x))^2 - 2\lambda \cdot \frac{1}{3}\lambda + \frac{1}{3}\lambda^2$$

$$= \int_0^1 (f(x))^2 - \frac{1}{3}\lambda^2$$

$f(x) = \sin\frac{\pi x}{n}$, $\lambda = 3\int_0^1 x \sin\frac{\pi x}{n} dx$

u	x	v'	$\sin\frac{\pi x}{n}$
u'	1	v	$\frac{-n}{\pi}\cos\frac{\pi x}{n}$

$$= -\frac{3n}{\pi}\left[x\cos\frac{\pi x}{n}\right]_0^1 + 3\frac{n}{\pi}\int_0^1 \cos\frac{\pi x}{n} dx$$

$$= \frac{-3n}{\pi}\cos\frac{\pi}{n} + 3\left(\frac{n}{\pi}\right)^2\left[\sin\frac{\pi x}{n}\right]_0^1$$

$$= 3\frac{n}{\pi}\left(\frac{n}{\pi}\sin\frac{\pi}{n} - \cos\frac{\pi}{n}\right)$$

As $n \to \infty$, $\pi/n \to 0$, so

$$\lambda \approx 3\frac{n}{\pi}\left(\frac{n}{\pi}\cdot\left(\frac{\pi}{n} - \frac{1}{6}\frac{\pi^3}{n^3}\right) - \left(1 - \frac{1}{2}\frac{\pi^2}{n^2}\right)\right)$$
$$= 3\frac{n}{\pi}\left(1 - \frac{1}{6}\frac{\pi^2}{n^2} - 1 + \frac{1}{2}\frac{\pi^2}{n^2}\right)$$

$$= 3\frac{n}{\pi}\left(\frac{1}{3}\frac{\pi^2}{n^2}\right)$$
$$= \pi/n$$

Clearly as $n \to \infty$, $\lambda^2 \approx \pi^2/n^2 \to 0$

Further, $\int_0^1 \sin^2(\pi x/n)\,dx = \left[\frac{1}{2}x - \frac{n\sin\frac{2\pi x}{n}}{4\pi}\right]_0^1$

$$= \frac{1}{2} - \frac{n}{4\pi}\sin\frac{2\pi}{n}$$

$$\approx \frac{1}{2} - \frac{n}{4\pi}\cdot\frac{2\pi}{n}$$
$$= \frac{1}{2} - \frac{1}{2}$$
$$= 0$$

So $\lim_{n\to\infty} R^2 = \int_0^1 \sin^2(\pi x/n) - \frac{1}{3}\lambda^2$
$$= 0 + 0$$
$$= 0$$

This is to be expected, as R^2 is the squared error when approximating $g(x)$ with a linear function. As $n \to \infty$, $\sin\frac{\pi x}{n}$ is stretched horizontally and on $[0,1]$ approaches a straight line, so the error $R^2 \to 0$.

STEP II 2000 Q6

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= \frac{2\frac{\sin\theta/2}{\cos\theta/2}}{1/\cos^2\theta/2} = \frac{2t}{\sec^2\theta/2} = \frac{2t}{1+t^2}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= \frac{1-\frac{\sin^2\theta/2}{\cos^2\theta/2}}{\sec^2\theta/2} = \frac{1-t^2}{1+t^2}$$

$$\frac{1+\cos\theta}{\sin\theta} = \frac{1+t^2+1-t^2}{1+t^2} \div \frac{2t}{1+t^2}$$

$$= \frac{2}{2t}$$

$$= \frac{1}{t}$$

$$= \frac{\cos\theta/2}{\sin\theta/2}$$

$$= \frac{\sin(\pi/2-\theta/2)}{\cos(\pi/2-\theta/2)}$$ using symmetry of graphs

$$= \tan(\pi/2 - \theta/2)$$

$$\int_0^{\pi/2} \frac{1}{1+\cos\alpha\sin\theta}\,d\theta$$

set $t = \tan\theta/2$

$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}(1+t^2)$$

$$= \int_0^1 \frac{1}{1+\cos\alpha\cdot\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{t^2+2t\cos\alpha+1}\,dt$$

$$= \int_0^1 \frac{2}{(t+\cos\alpha)^2+1-\cos^2\alpha}\,dt$$

$$= \int_0^1 \frac{2}{(t+\cos\alpha)^2+\sin^2\alpha}\,dt$$

$$= 2\left[\frac{\arctan\left(\frac{t+\cos\alpha}{\sin\alpha}\right)}{\sin\alpha}\right]_0^1$$

$$= \frac{2}{\sin\alpha}\left(\arctan\left(\frac{1+\cos\alpha}{\sin\alpha}\right) - \arctan\left(\frac{\cos\alpha}{\sin\alpha}\right)\right)$$

$$= \frac{2}{\sin\alpha}\left(\arctan(\tan(\pi/2-\alpha/2)) - \arctan(\tan(\pi/2-\alpha))\right)$$

$$= \frac{2}{\sin\alpha}\left(\pi/2-\alpha/2-\pi/2+\alpha\right)$$

$= \dfrac{\alpha}{\sin\alpha}$, as required.

Now consider $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin\alpha\cos\theta}\,d\theta$

Make the substitutions $\beta = \pi/2-\alpha$, $\phi = \pi/2-\theta$, $d\phi = -d\theta$

$$= -\int_{\pi/2}^{0} \frac{1}{1+\cos\beta\sin\phi}\,d\phi$$

$$= \int_0^{\pi/2} \frac{1}{1+\cos\beta\sin\phi}\,d\phi$$

$$= \beta/\sin\beta$$

$$= \frac{\pi/2-\alpha}{\cos\alpha}$$

STEP II 2000 Q7

$$\underline{r} = \lambda \begin{pmatrix} \cos\theta + \sqrt{3} \\ \sqrt{2}\sin\theta \\ \cos\theta - \sqrt{3} \end{pmatrix} \quad \text{and} \quad \underline{r}' = \mu \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We have $\underline{x} \cdot \underline{y} = |\underline{x}||\underline{y}|\cos\varphi$

$$\Rightarrow \cos\varphi = \frac{\underline{x} \cdot \underline{y}}{|\underline{x}||\underline{y}|}$$

So the angle between the lines is

$$\varphi = \arccos\left(\frac{a(\cos\theta + \sqrt{3}) + b\sqrt{2}\sin\theta + c(\cos\theta - \sqrt{3})}{\sqrt{a^2+b^2+c^2} \cdot \sqrt{\cos^2\theta + 3 - \cancel{2\sqrt{3}\cos\theta} + 2\sin^2\theta + \cos^2\theta + 3 + \cancel{2\sqrt{3}\cos\theta}}} \right)$$

$$= \arccos\left(\frac{(a+c)\cos\theta + b\sqrt{2}\sin\theta + \sqrt{3}(a-c)}{\sqrt{a^2+b^2+c^2} \cdot \sqrt{8}} \right)$$

For the angle to be $\pi/6$, we require

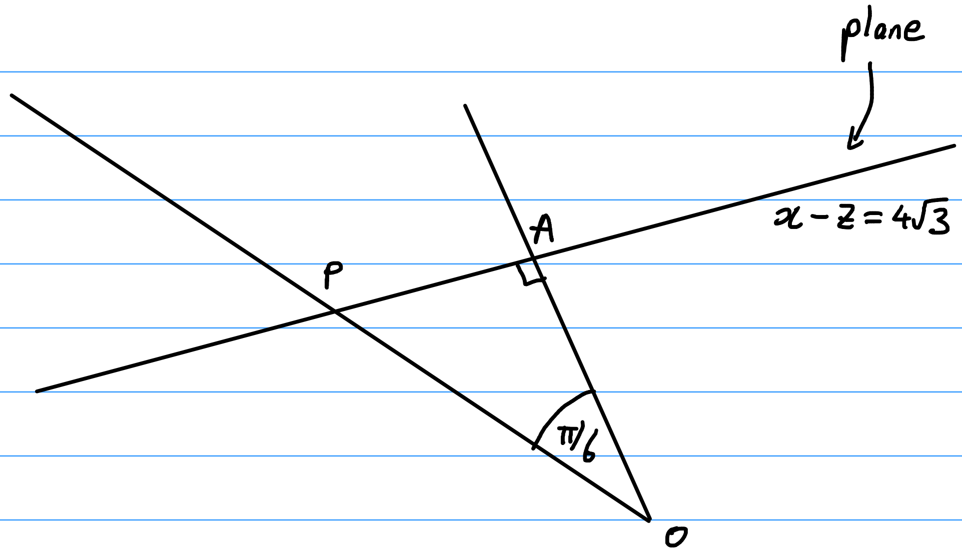
$$\frac{(a+c)\cos\theta + b\sqrt{2}\sin\theta + \sqrt{3}(a-c)}{\sqrt{a^2+b^2+c^2} \cdot \sqrt{8}} = \frac{\sqrt{3}}{2}$$

Set $a = -c$ and $b = 0$ gets

$$\frac{2\cancel{\sqrt{3}}\cancel{c}}{\sqrt{2}\cancel{c}\sqrt{8}} = \frac{\cancel{\sqrt{3}}}{2}$$

$\Rightarrow \frac{2}{4} = \frac{1}{2}$ which is true.

So setting $\underline{r}' = \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ works.



Note the normal to the plane is $\begin{pmatrix}1\\0\\-1\end{pmatrix}$. Hence as θ varies, the line makes an angle of $\pi/6$ with this normal. As this angle is constant, PA is constant, so P describes a circle.

The centre is when $x - z = 4\sqrt{3} \Rightarrow \mu - (-\mu) = 4\sqrt{3}$
$\Rightarrow \mu = 2\sqrt{3}$

So $A = \begin{pmatrix}2\sqrt{3}\\0\\-2\sqrt{3}\end{pmatrix}$

Setting $\theta = \pi/2$ gives $\underline{r} = \lambda\begin{pmatrix}\sqrt{3}\\\sqrt{2}\\-\sqrt{3}\end{pmatrix}$. As $x - z = 4\sqrt{3}$, we have $2\sqrt{3}\lambda = 4\sqrt{3}$
$\Rightarrow \lambda = 2.$

Then $P = \begin{pmatrix}2\sqrt{3}\\2\sqrt{2}\\-2\sqrt{3}\end{pmatrix}$. So $AP = \begin{pmatrix}0\\2\sqrt{2}\\0\end{pmatrix}$ so radius is $2\sqrt{2}$.

STEP II 2000 Q8

(i) $\frac{dy}{dx} + 4xe^{-x^2}(y+3)^{1/2} = 0$

$$\Rightarrow \int \frac{1}{\sqrt{y+3}}\,dy = \int -4xe^{-x^2}\,dx$$

$$\Rightarrow 2\sqrt{y+3} = 2e^{-x^2} + c$$

$$\Rightarrow \sqrt{y+3} = e^{-x^2} + A$$

$$\Rightarrow y = (A + e^{-x^2})^2 - 3$$

$y(0) = 6 \Rightarrow 6 = (A+1)^2 - 3$

$\Rightarrow (A+1)^2 = 9$

$\Rightarrow A = 2$

So $y = (2 + e^{-x^2})^2 - 3$

As $x \to \infty$, $y \to (2+0)^2 - 3$

$= 4 - 3$

$= 1$

(ii) $\frac{dy}{dx} - xe^{6x^2}(y+3)^{1-k} = 0$

$$\Rightarrow \int (y+3)^{k-1}\,dy = \int xe^{6x^2}\,dx$$

$$\Rightarrow \frac{1}{k}(y+3)^k = \frac{1}{12}e^{6x^2} + c \qquad (k \neq 0)$$

$$\Rightarrow (y+3)^k = A + \frac{k}{12}e^{6x^2}$$

$$\Rightarrow y = \left(A + \frac{k}{12}e^{6x^2}\right)^{1/k} - 3$$

$$\Rightarrow ye^{-3x^2} = \left(A + \frac{k}{12}e^{6x^2}\right)^{1/k}\left(e^{-3kx^2}\right)^{1/k} - 3e^{-3x^2}$$

$$= \left(Ae^{-3kx^2} + \frac{k}{12}e^{x^2(6-3k)}\right)^{1/k} - 3e^{-3x^2}$$

$k \geqslant 0$ $\qquad$ $6-3k=0$

$\Rightarrow k=2$

Then $ye^{-3x^2} \longrightarrow \left(0+\frac{2}{12}\right)^{1/2}$

$$= \frac{1}{\sqrt{6}}$$

STEP II 2000 Q9

Jane: $\frac{dv}{dt} = g - kv$

$\Rightarrow \int \frac{1}{g-kv}\,dv = \int dt$

$\Rightarrow -\frac{1}{k}\ln|g-kv| = t + c$

$\Rightarrow \ln|g-kv| = -kt + c'$

$\Rightarrow g - kv = Ae^{-kt}$

$\Rightarrow v = \frac{1}{k}(g - Ae^{-kt})$

$v(0) = 0 \Rightarrow A = g$

So $v = \frac{g}{k}(1 - e^{-kt}) \neq g/k$

Karen: $\frac{dv}{dt} = g - kv - \frac{2k^2}{g}v^2$

$\Rightarrow \int \frac{1}{g - kv - \frac{2k^2}{g}v^2}\,dv = \int dt$

$\Rightarrow \int \frac{-1}{(\frac{2k}{g}v - 1)(kv + g)} = t + c$

$\Rightarrow \frac{-g}{k^2}\int \frac{1}{(2v - g/k)(v + g/k)}\,dv = t + C$

Now $\frac{1}{(2v - g/k)(v + g/k)} = \frac{A}{2v - g/k} + \frac{B}{v + g/k}$

$\Rightarrow A(v + g/k) + B(2v - g/k) \equiv 1$

Setting $v = -g/k \Rightarrow B = -k/3g$

$v = g/2k \Rightarrow A = 2k/3g$

So $\frac{-g}{k^2}\frac{1}{(2v - g/k)(v + g/k)} = \frac{-1}{3k}\left(\frac{2}{2v - g/k} - \frac{1}{v + g/k}\right)$

Hence, integrating,

$$\frac{-1}{3k}\left(\ln|2v-g/k| - \ln|v+g/k|\right) = t + c$$

$$\Rightarrow \ln\left|\frac{v+g/k}{2v-g/k}\right| = 3kt + c'$$

$$\Rightarrow \frac{v+g/k}{2v-g/k} = Ae^{3kt}$$

$v(0)=0 \Rightarrow -1 = A$

So $v + g/k = -2ve^{3kt} + g/k\, e^{3kt}$

$$\Rightarrow v(1+2e^{3kt}) = g/k(e^{3kt}-1)$$

$$\Rightarrow v = \frac{g(e^{3kt}-1)}{k(1+2e^{3kt})}$$

$$= \frac{g(1-e^{-3kt})}{k(2+e^{-3kt})}$$

As $t \to \infty$, $v \to \frac{g}{2k}$, but for $t<\infty$, $v < \frac{g}{2k}$ (numerator $< g$, denominator $> 2k$)

For Jane, $\frac{g}{k}(1-e^{-kt}) = \frac{g}{3k}$

$$\Rightarrow 1-e^{-kt} = \frac{1}{3}$$

$$\Rightarrow e^{-kt} = 2/3$$

$$\Rightarrow -kt = \ln(2/3)$$

$$\Rightarrow t = k^{-1}\ln(3/2)$$

For karen, $\frac{1-e^{-3kt}}{2+e^{-3kt}} = \frac{1}{3}$

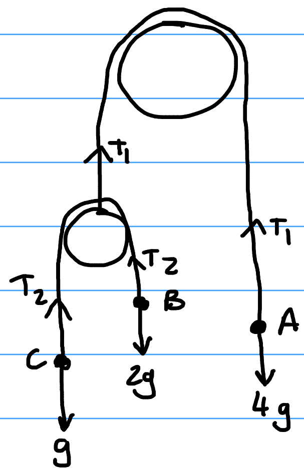
$$\Rightarrow 3-3e^{-3kt} = 2+e^{-3kt}$$

$$\Rightarrow 1 = 4e^{-3kt}$$

$$\Rightarrow 3kt = \ln 4$$

$$\Rightarrow t = (3k)^{-1}\ln 4$$

STEP II 2000 Q10



Initially, A is fixed. Considering B and C, B moves down and C moves up with common acceleration

$B: 2g - T_2 = 2a$

$A: T_2 - g = a$

So $3g = a$

$\Rightarrow a = g/3, \; T_2 = 4g/3$

Now, A is released. Let a_1 be the downward acceleration of A, and let a_2 be the downward acceleration of B wrt its pulley. Then

$A: \quad 4g - T_1 = 4a_1 \qquad (1)$

Pulley: $T_1 - 2T_2 = 0 \Rightarrow T_1 = 2T_2 \qquad (2)$

$B: \; 2g - T_2 = 2(a_2 - a_1) \qquad (3)$

$C: \; T_2 - g = a_1 + a_2 \qquad (4)$

Substituting (2) into (1) gives $2g - T_2 = 2a_1$

$\Rightarrow T_2 = 2(g - a_1)$

Substituting this into ③ and ④ gives

$$2g - 2(g-a_1) = 2(a_2 - a_1)$$
$$\Rightarrow \quad a_1 = a_2 - a_1$$
$$\Rightarrow \quad 2a_1 = a_2$$

$$2(g-a_1) - g = a_1 + a_2$$
$$\Rightarrow \quad g - 2a_1 = a_1 + a_2$$
$$\Rightarrow \quad 3a_1 + a_2 = g$$

Combining these, $3a_1 + 2a_1 = g$
$$\Rightarrow a_1 = g/5, \; a_2 = 2g/5$$

So A has acceleration $g/5$ downwards, B has acceleration $g/5$ downwards, and C has acceleration $3g/5$ upwards.

At the end of the first phase of motion, B has speed $gT/3$ and has moved $\frac{1}{2}at^2 = gT^2/6$.

So, when A is released, B needs to trave through a further $gT^2\left(\frac{3}{5} - \frac{1}{6}\right) = \frac{13}{30}gT^2$.

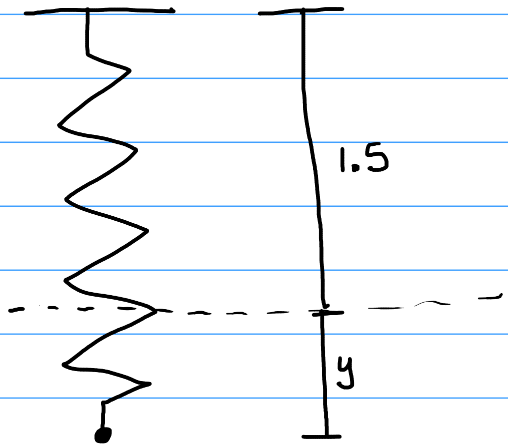
s	$\frac{13}{30}gT^2$
u	$gT/3$
v	$\times$
a	$g/5$
t	$?$

$$\frac{13}{30}gT^2 = \frac{gT}{3}t + \frac{g}{10}t^2$$
$$\Rightarrow 3t^2 + 10Tt - 13T^2 = 0$$
$$\Rightarrow (3t + 13T)(t - T) = 0$$
$$\Rightarrow t = T \quad (t > 0).$$

So after A is released, the particles move for a further T seconds.

Hence the final speed of A is $\frac{1}{5}gT$.

STEP II 2000 Q11



Let the speed of the particle at equilibrium be v. Then, by considering energy,

$$\frac{5g\times 0.\cancel{5}^2}{2\times 1.5} = \cancel{0}.5\times g\times \cancel{0}.5 + 0.\cancel{5}^2\times v^2$$

EPE = GPE + KE

$$\Rightarrow v^2 = g\left(\tfrac{5}{3}-1\right)$$
$$= \tfrac{2}{3}g > 0.$$

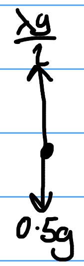
As $v^2>0$, the particle reaches this point. Hence the string becomes slack.

Instead, considering the speed at the ceiling,

$$\frac{5g\times 0.5^2}{2\times 1.5} = 0.5\times g\times 2 + 0.5^2\times v^2$$

$$\Rightarrow v^2 = g\left(\tfrac{5}{3}-4\right) < 0.$$ So this is not possible.

When the string is in tension,



The particle is in equilibrium when $0.5g = \frac{5gy}{1.5}$
$\Rightarrow y = 0.15$

Otherwise, the restoring force is

$$T - 0.5g = \frac{5g}{1.5}(0.15 + d) - 0.5g$$

where d is displacement from equilibrium

$$= \frac{5g}{1.5}d$$
$$= \frac{10}{3}d \quad , \text{ is SHM.}$$

So, $d'' = \frac{-10}{3 \times 0.5}d$
$= -\frac{20}{3}d$

This is SHM with period $2\pi\sqrt{\frac{3}{20g}}$ and amplitude $0.5 - 0.15 = 0.35\,m$

The particle takes $\frac{2\pi}{4}\sqrt{\frac{3}{20g}}$ from the start to equilibrium. Then to reach the point where EPE=0,

$$0.15 = 0.35 \sin\sqrt{\frac{20g}{3}}\,t$$

$$\Rightarrow t = \sqrt{\frac{3}{20g}}\sin^{-1}\left(\tfrac{3}{7}\right)$$

At this point, $v^2 = \omega^2(a^2 - x^2)$

$$= \frac{20g}{3}\left(0.35^2 - 0.15^2\right)$$

$$= \frac{20g}{3}\left(\frac{49-9}{400}\right)$$

$$= \frac{20g}{3}\left(\frac{1}{10}\right)$$

$$= \frac{2g}{3}$$

The particle then moves freely under gravity.

s 0

u $\sqrt{2g/3}$

v $-\sqrt{2g/3}$

a $-g$

t $?$

So $-\sqrt{2g/3} = \sqrt{2g/3} - gt$

$$\Rightarrow t = \frac{2\sqrt{2g/3}}{g}$$

$$= \sqrt{\frac{8}{3g}}$$

So, the total time is

$$\frac{2\pi}{4}\sqrt{\frac{3}{20g}} + \sin^{-1}\left(\frac{3}{7}\right)\sqrt{\frac{3}{20g}} + \sqrt{\frac{8}{3g}} + \sin^{-1}\left(\frac{3}{7}\right)\sqrt{\frac{3}{20g}} + \frac{2\pi}{4}\sqrt{\frac{3}{20g}}$$

$$= \frac{\pi}{2}\sqrt{\frac{3}{5g}} + \sin^{-1}\left(\frac{3}{7}\right)\sqrt{\frac{3}{5g}} + \sqrt{\frac{8}{3g}}$$

$$= \sqrt{\frac{8}{3g}} + \sqrt{\frac{3}{5g}}\left(\pi/2 + (\pi/2 - \arccos(3/7)\right)$$

$= \left(\frac{8}{3g}\right)^{1/2} + \left(\frac{3}{5g}\right)^{1/2}(\pi - \arccos(3/7))$, as required.

STEP II 2000 Q12

If he takes the king's Cross train, his journey is

$$N(55, 25) + N(30, 144) = N(85, 13^2)$$

For Liverpool Street, it is

$$5 + N(65, 16) + N(25, 9) = N(95, 5^2)$$

$P(KX > 105) = P(Z > \frac{20}{13})$
$P(LS > 105) = P(Z > \frac{10}{5})$
$20/13 < 10/5$, so $P(KX \text{ late}) > P(LS \text{ late})$.

Now $P(\text{late}) = pA + (1-p)B$
So $E(\# \text{ of late}) = M(pA + (1-p)B)$
$= p(A-B)M + BM$
This is linear and increasing in p, so has minimum value BM $(p=0)$ and maximum value AM $(p=1)$.
So $BM \leq L \leq AM$

Now $P(KX | \text{Late}) = \frac{3}{5}$
$\Rightarrow \frac{P(KX \cap \text{Late})}{P(\text{Late})} = \frac{3}{5}$

$\Rightarrow \frac{pA}{pA + (1-p)B} = \frac{3}{5}$

$\Rightarrow 5Ap = 3Ap + 3B - 3Bp$
$\Rightarrow p(2A + 3B) = 3B$

$\Rightarrow p = \frac{3B}{2A + 3B}$

STEP II 2000 Q13

$$P_N = \binom{200}{11}\left(\frac{200}{N}\times\frac{199}{N-1}\times\cdots\times\frac{190}{N-10}\right)\left(\frac{N-200}{N-11}\times\frac{N-201}{N-12}\times\cdots\times\frac{N-388}{N-199}\right)$$

ways of picking the 11 recaptured

probability of choosing these 11

probability of choosing the others

$$= \left(\frac{200!}{11!\,189!}\right)\left(\frac{200!}{189!}\times\frac{(N-200)!}{(N-389)!}\times\frac{(N-200)!}{N!}\right)$$

$$= \frac{(200!)^2}{11!\,(189)!^2}\times\frac{((N-200)!)^2}{N!\,(N-389)!}$$

Suppose $P_N > P_{N-1} \Leftrightarrow \frac{((N-200)!)^2}{N!\,(N-389)!} > \frac{((N-201)!)^2}{(N-1)!\,(N-390)!}$

$$\Leftrightarrow \frac{(N-200)^2}{N(N-389)} > 1$$

$$\Leftrightarrow N^2-400N+40{,}000 > N^2-389N$$

$$\Leftrightarrow 11N < 40{,}000$$

$$\Leftrightarrow N < \frac{40{,}000}{11}$$

So, if $N < \frac{40000}{11}$, then $P_N > P_{N-1}$, otherwise if $N > \frac{40000}{11}$, then $P_N < P_{N-1}$

$$11\overline{)\,40{,}000}\quad 3636.\cdots$$

So $N = 3636$ gives P_N maximal.

Now 389 votes are marked.

The probability of exactly j marked voles is

$$\frac{\binom{389}{j}\binom{N-389}{200-j}}{\binom{N}{200}}$$

Because these probabilities must sum to 1,

$$\sum_{j=0}^{200} \binom{389}{j}\binom{N-389}{200-j} = \binom{N}{200}$$

Substituting $N=3636$ gets

$$\sum_{j=0}^{200} \binom{389}{j}\binom{3247}{200-j} = \binom{3636}{200},$$ as required.

STEP II 2000 Q14

we have $\int_0^1 ky^n(1-y)^n \, dy = 1$

$$\Rightarrow k \cdot \frac{n!\,n!}{(2n+1)!} = 1$$

$$\Rightarrow k = \frac{(2n+1)!}{(n!)^2}$$

$$EY = k\int_0^1 y^{n+1}(1-y)^n dy$$

$$= k \cdot \frac{(n+1)!\,n!}{(2n+2)!}$$

$$= \frac{(2n+1)!\,(n+1)!\,n!}{(n!)^2(2n+2)!}$$

$$= \frac{n+1}{2n+2}$$

$$= \frac{1}{2}$$

$$EY^2 = k\int_0^1 y^{n+2}(1-y)^n \, dy$$

$$= k\,\frac{(n+2)!\,n!}{(2n+3)!}$$

$$= \frac{(2n+1)!\,(n+2)!\,n!}{(n!)^2(2n+3)!}$$

$$= \frac{(n+1)(n+2)}{(2n+2)(2n+3)}$$

$$= \frac{n+2}{2(n+3)}$$

So $\operatorname{Var} Y = \frac{n+2}{2(2n+3)} - \frac{1}{4}$

$$= \frac{2n+4-(2n+3)}{4(2n+3)}$$

$$= \frac{1}{4(2n+3)}$$

For large n, $Y \sim N\left(\frac{1}{2}, \frac{1}{4(2n+3)}\right)$

$\bar{X} \sim N\left(\frac{1}{2}, \frac{1}{12(2n+1)}\right)$

Now $\frac{1}{4(2n+3)} > \frac{1}{12(2n+1)}$

$\Rightarrow 24n+12 > 8n+12$

$\Rightarrow n>0$

So $\operatorname{Var} Y > \operatorname{Var} X$

Hence we have $P(|Y-1/2| < d/\sqrt{n}) < P(|\bar{X}-1/2| < d/\sqrt{n})$