STEP I 2000 Q1

(i) $\log_{10}5 = \log_{10}\frac{10}{2}$

$= \log_{10}10 - \log_{10}2$

$= 1 - 0.301$

$= 0.699$

$\log_{10}6 = \log_{10}(2\times3)$

$= \log_{10}2 + \log_{10}3$

$= 0.301 + 0.477$

$= 0.778$

Now $5\times10^{47} < 3^{100} < 6\times10^{47}$

$\Leftrightarrow$ $\log_{10}5 + 47 < 100\log_{10}3 < \log_{10}6 + 47$ (because log is strictly increasing)

$\Leftrightarrow$ $47.699 < 47.771 < 47.778$

which is true.

So the first digit of 3^{100} is 5.

(ii) Note $\log_{10}2^{1000} = 1000\log_{10}2$

$= 301.0299\ldots$

$301 < 301.0299 < 301.30102996$

$10^{301} < 2^{1000} < 2\times10^{301}$

So the first digit of 2^{1000} is 1.

$\log_{10}2^{10,000} = 3010.29996$

$3010 < 3010.29996 < 3010.3010\ldots$

$10^{3010} < 2^{10,000} < 2\times10^{3010}$

So the first digit of $2^{10,000}$ is 1.

$\log_{10} 2^{100,000} = 30,102.99996$

Note $\log_{10} 9 = 2\log_{10} 3$
≈ 0.854

$$30102.854 < 30102.99996 < 30103$$
$$9 \times 10^{30102} < 2^{100,000} < 10^{30103}$$

So the first digit of $2^{100,000}$ is 9.

STEP I 2000 Q2

$$\left(x^4-\frac{1}{x^2}\right)^5\left(x-\frac{1}{x}\right)^6$$
$$=\left(\frac{1}{x^2}\right)^5(x^6-1)^5\left(\frac{1}{x}\right)^6(x^2-1)^6$$

$$=\frac{1}{x^{16}}(x^6-1)^5(x^2-1)^6$$

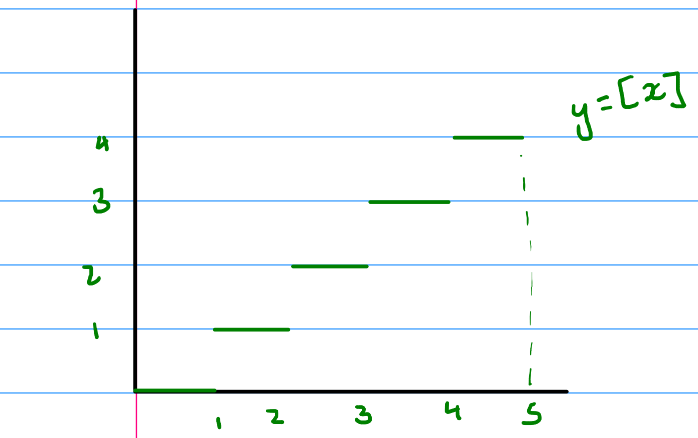
For the x^{-12} term, we need x^4 from $(x^6-1)^5(x^2-1)^6$, which can only come from the $(-1)^5(x^2)^2(-1)^4$ term which has coefficient $\binom{5}{0}(-1)^5\binom{6}{2}(-1)^4=-15$.

For the x^2 term, we need the x^{18} term from $(x^6-1)^5(x^2-1)^6$, which comes from the $(x^6)^3$, $(x^6)^2(x^2)^3$, and $(x^6)(x^2)^6$ terms. Thus the coefficient (noting all of these are positive) is $\binom{5}{3}\binom{6}{0}+\binom{5}{2}\binom{6}{3}+\binom{5}{1}\binom{6}{6}=10\times1+10\times20+5\times1=215$.

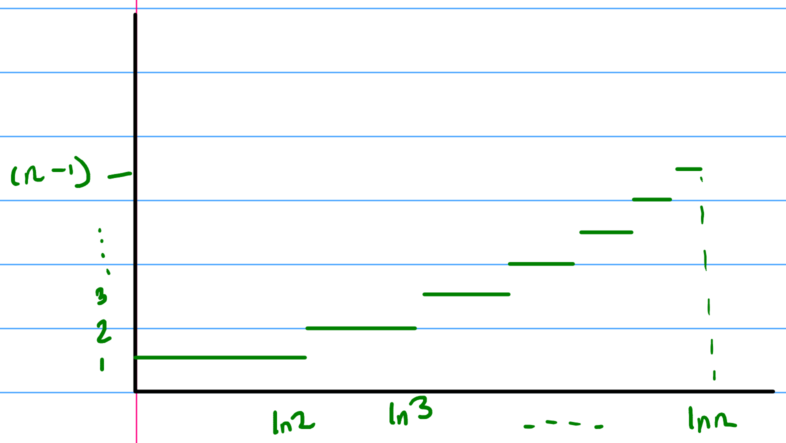
Note $(x^2-1)^{11}(x^4+x^2+1)^5$
$$=(x^2-1)^6(x^2-1)^5\cdot(x^4+x^2+1)^5$$
$$=(x^2-1)^6(x^6+x^4+x^2-x^4-x^2-1)^5$$
$$=(x^2-1)^6(x^6-1)^5$$

We previously worked out the coefficient of x^4 in this expansion is -15.
To get x^{38}, we must have the $(x^2)^4(x^6)^5$ term, with coefficient $\binom{6}{4}\binom{5}{5}=15$.

STEP I 2000 Q3



$$\int_0^5 [x]\,dx = 0+1+2+3+4$$
$$=10$$



$$\int_0^{\ln(n)} [e^x]\,dx = \ln 2 + 2(\ln 3 - \ln 2) + 3(\ln 4 - \ln 3) + \text{———} + (n-1)(\ln(n) - \ln(n-1))$$
$$= -\ln 2 - \ln 3 - \ln 4 + \text{———} - \ln(n-1) - \ln(n) + n\ln(n)$$
$$= -\ln(1\times 2\times 3\times \text{———} \times n) + n\ln(n)$$
$$= n\ln(n) - \ln(n!)$$

STEP I 2000 Q4

1(i) The largest value occurs either at a stationary point, or at $x=0$ or $x=1$.

$$y = \frac{x^6}{(x^2+1)^4}$$

$$\frac{dy}{dx} = \frac{6x^5(x^2+1)^4 - 8x(x^2+1)^3(x^6)}{(x^2+1)^8}$$

$$= \frac{2x^5[3(x^2+1) - 4x^2]}{(x^2+1)^5}$$

$$= \frac{2x^5(3-x^2)}{(x^2+1)^5} = 0 \text{ at } x = \pm\sqrt{3} \notin [0,1].$$

So the maximum occurs at an endpoint.

$y(0) = 0 \quad y(1) = \frac{1}{16}$. So the largest value is $\frac{1}{16}$.

1(ii) $$\frac{d}{dx}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) = \frac{(5Ax^4+3Bx^2+C)(x^2+1)^3 - 6x(x^2+1)^2(Ax^5+Bx^3+Cx)}{(x^2+1)^6}$$

$$= \frac{(5Ax^4+3Bx^2+C)(x^2+1) - 6x(Ax^5+Bx^3+Cx)}{(x^2+1)^4}$$

$$= \frac{5Ax^6+5Ax^4+3Bx^4+3Bx^2+Cx^2+C-6Ax^6-6Bx^4-6Cx^2}{(x^2+1)^4}$$

$$= \frac{x^6(-A)+x^4(5A-3B)+x^2(3B-5C)+C}{(x^2+1)^4}$$

So $$\frac{d}{dx}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \frac{Dx^6}{(x^2+1)^4}$$

$$= \frac{1}{(x^2+1)^4}\left[(D-A)x^6+(5A-3B)x^4+(3B-5C)x^2+C\right] \equiv \frac{1}{(x^2+1)^4}$$

$\Rightarrow C=1$

$\Rightarrow B=5/3$

$\Rightarrow A=1 \qquad \Rightarrow D=1$

So, $\int_0^1 \frac{1}{(x^2+1)^4}\,dx = \left[\frac{x^5+\frac{5}{3}x^3+x}{(x^2+1)^3}\right]_0^1 + \int_0^1 \frac{x^6}{(x^2+1)^4}\,dx$

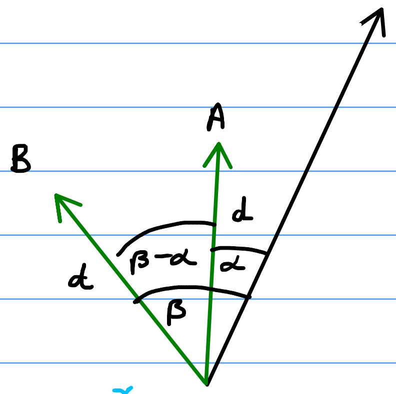
$$= \frac{11/3}{8} + \int_0^1 \frac{x^6}{(x^2+1)^4}\,dx$$

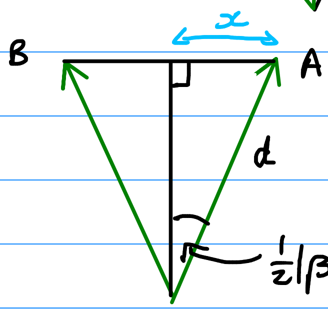
$$= {}^{11}/_{24} + \int_0^1 \frac{x^6}{(x^2+1)^4}\,dx$$

And, because $0 \leq \frac{x^6}{(x^2+1)^4} \leq \frac{1}{16}$ for $x \in [0,1]$, we have $0 \leq \int_0^1 \frac{x^6}{(x^2+1)^4}\,dx \leq \frac{1}{16}$, and so

$$\frac{11}{24} \leq \int_0^1 \frac{1}{(x^2+1)^4}\,dx \leq \frac{11}{24} + \frac{1}{16}$$

STEP I 2000 Q5



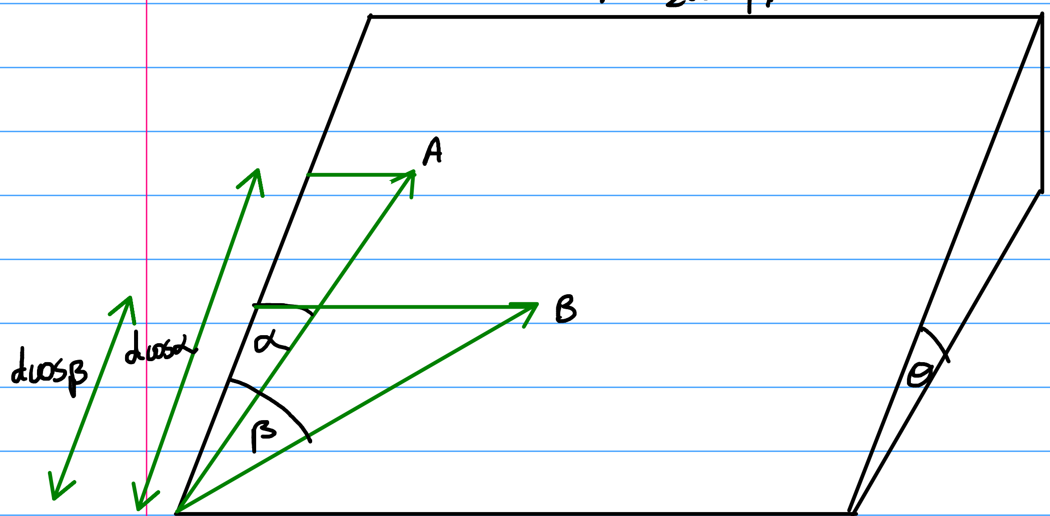


$\frac{1}{2}|\beta-\alpha|$, the modulus because we do not now if $\beta>\alpha$ or $\alpha>\beta$.

$x = d\sin\frac{1}{2}|\beta-\alpha|$, so $AB = 2d\sin\frac{1}{2}|\beta-\alpha|$

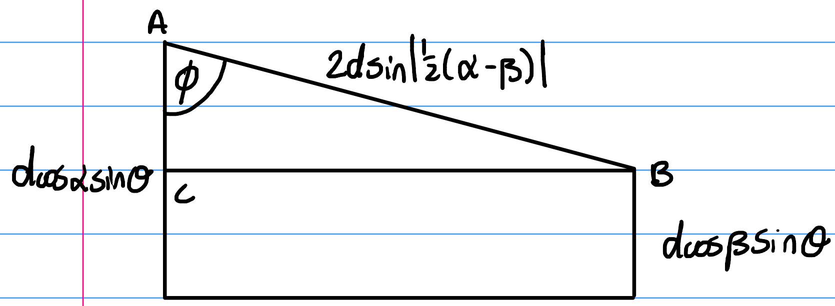
$= 2d|\sin\frac{1}{2}(\beta-\alpha)|$

$= 2d|\sin\frac{1}{2}(\alpha-\beta)|$



The height of A above the ground is $d\cos\alpha\sin\theta$ and of B is $d\cos\beta\sin\theta$.

Viewed horizontally,



Considering triangle ABC, we have

$$\cos\phi = \frac{|d\cos\alpha\sin\theta - d\cos\beta\sin\theta|}{2d\sin|\frac{1}{2}(\alpha-\beta)|}$$

$$= \frac{\sin\theta|\cos\alpha - \cos\beta|}{2|\sin\frac{1}{2}(\alpha-\beta)|}$$

$$= \frac{\sin\theta|2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)|}{2|\sin\frac{1}{2}(\alpha-\beta)|}$$

$$= \frac{\sin\theta\sin\frac{1}{2}(\alpha+\beta)|\sin\frac{1}{2}(\alpha-\beta)|}{|\sin\frac{1}{2}(\alpha-\beta)|}$$

$= \sin\theta\sin\frac{1}{2}(\alpha+\beta)$, as required.

STEP I 2000 Q6

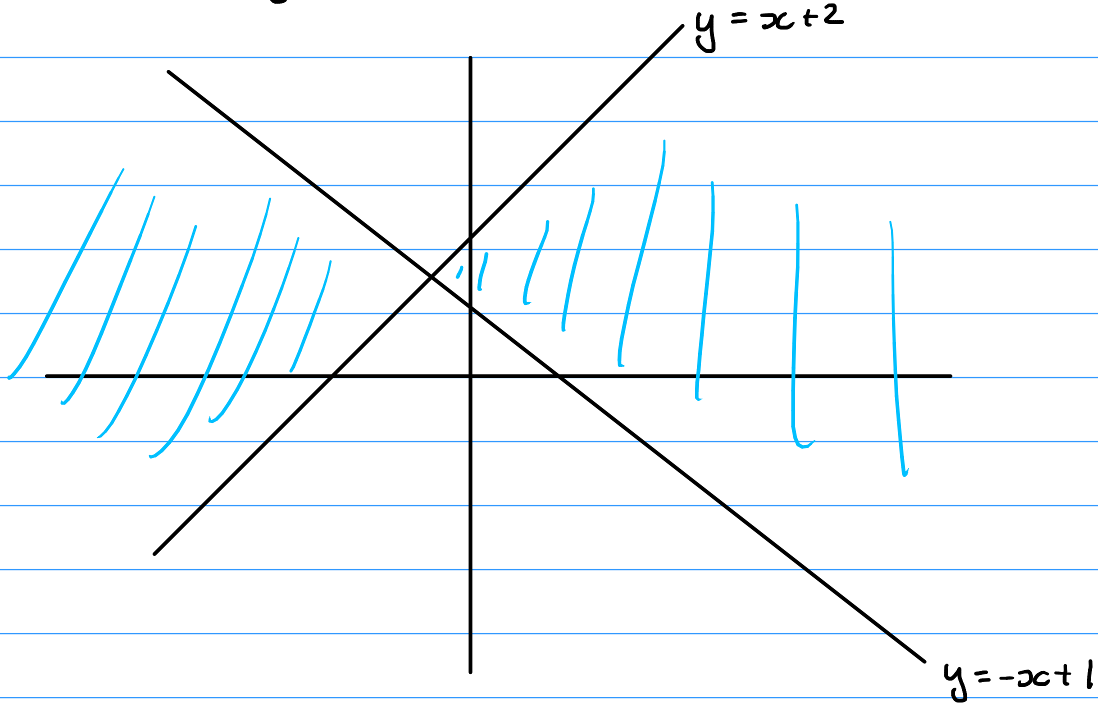
$(x-y+2)(x+y-1)$

$= x^2 + \cancel{xy} - x - \cancel{xy} - y^2 + y + 2x + 2y - 1$

$= x^2 - y^2 + x + 3y - 2$

So $x^2 - y^2 + x + 3y > 2$

$\iff (x-y+2)(x+y-1) > 0$



The shaded area satisfies the inequality.

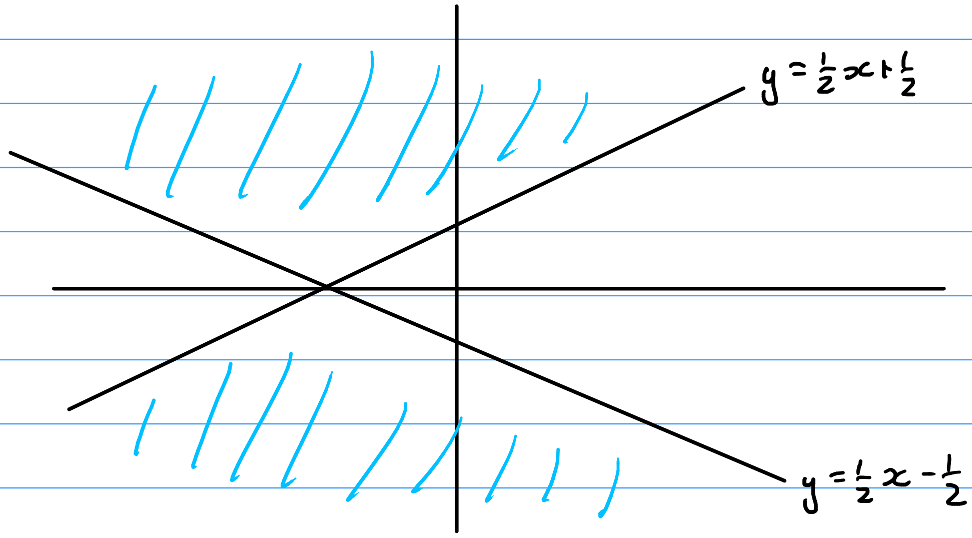
Now consider $(x-2y+1)(x+2y+2)$

$= x^2 + 2xy + 2x - 2xy - 4y^2 - 4y + x + 2y + 2$

$= x^2 - 4y^2 + 3x - 2y + 2$

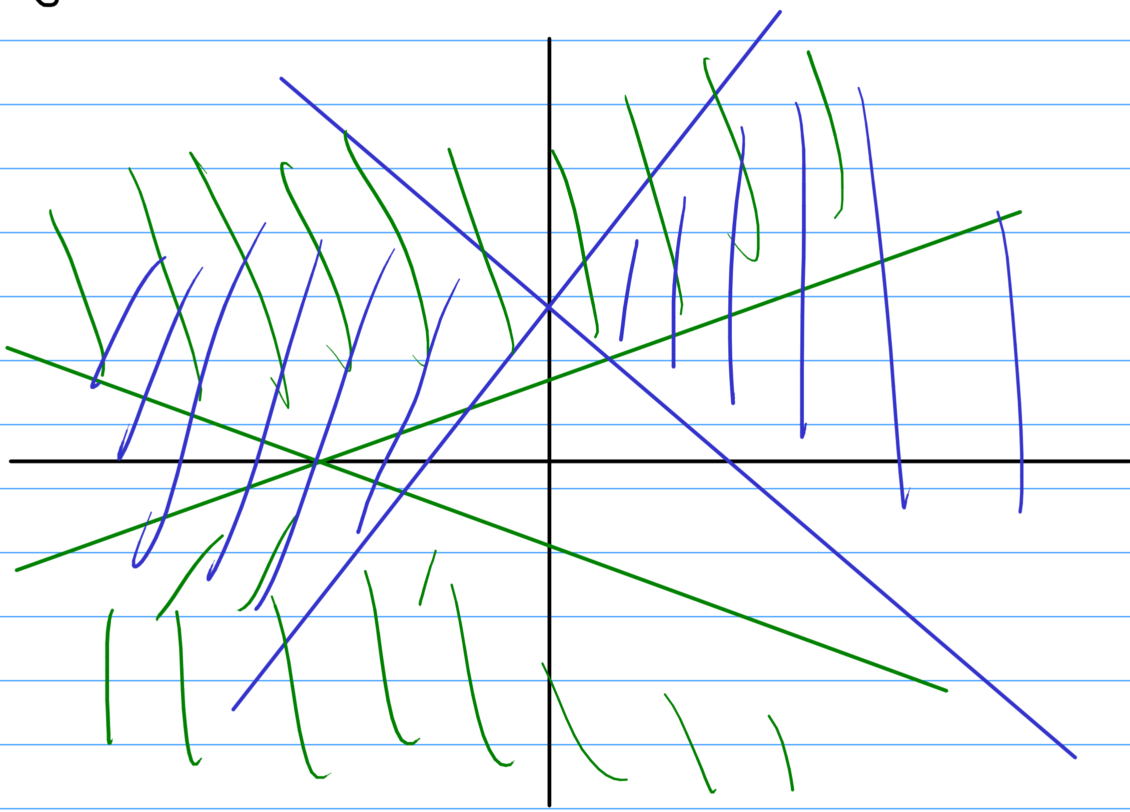
So $x^2 - 4y^2 + 3x - 2y < -2$

$\iff (x-y+2)(x+y-1) < 0$



The shaded area satisfies the inequality.

Putting both on the same axes,



Lots of points satisfy both inequalities, for example (2,2). Then

$$x^2 - y^2 + x + 3y = 8 > 2 \checkmark$$
$$x^2 - 4y^2 + 3x - 2y = -10 < -2 \checkmark$$

STEP I 2000 Q7

$$f(x) = ax - \frac{x^3}{1+x^2}$$

$$f'(x) = a - \frac{3x^2(1+x^2) - 2x(x^3)}{(1+x^2)^2}$$

$$= a - \frac{3x^2 + 3x^4 - 2x^4}{(1+x^2)^2}$$

$$= a + \frac{-x^4 - 3x^2}{(1+x^2)^2}$$

So, $a \geq 9/8 \Rightarrow f'(x) \geq 0$ is equivalent to showing $\frac{-x^4-3x^2}{(1+x^2)^2} \geq -9/8$ for all x.

Let $g(x) = \frac{-x^4-3x^2}{(1+x^2)^2}$

Then $g'(x) = \frac{(-4x^3-6x)(1+x^2)^2 + 4x(x^4+3x^2)(1+x^2)}{(1+x^2)^2} = 0$

$$\Leftrightarrow \quad 4x^5 + 12x^3\cancel{(1+x^2)} - (4x^3+6x)(1+x^2)^{\cancel{2}} = 0$$

$$\Leftrightarrow \quad x(4x^4 + 12x^2 - 4x^2 - 4x^4 - 6 - 6x^2) = 0$$

$$\Leftrightarrow \quad x(2x^2 - 6) = 0$$

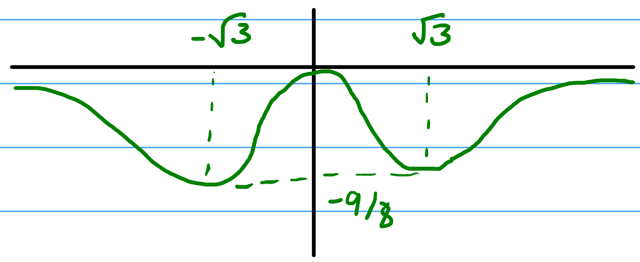
$$\Leftrightarrow \quad x = 0, \sqrt{3}, \text{ or } -\sqrt{3}$$

Now $g(0) = 0$

$g(\pm\sqrt{3}) = -\left(\frac{9+9}{16}\right)$

$= -9/8$

$\lim_{x \to \pm\infty} g(x) = 0$



So $g(x) \geq -9/8$ for all x, so we have proved the required result.

STEP I 2000 Q8

$$\int_{-1}^{1} |xe^x|\,dx = \int_{-1}^{0} |xe^x|\,dx + \int_{0}^{1} |xe^x|\,dx$$

$$= -\int_{-1}^{0} xe^x\,dx + \int_{0}^{1} xe^x\,dx$$

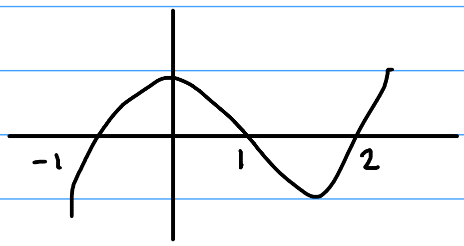
Note $\int xe^x\,dx = xe^x - e^x$

$$= -\left[xe^x - e^x\right]_{-1}^{0} + \left[xe^x - e^x\right]_{0}^{1}$$

$$= -(-1+2e^{-1}) + (1)$$

$$= 2 - 2e^{-1}$$

If $f(x) = x^3 - 2x^2 - x + 2$, then $f(1) = f(2) = f(-1) = 0$, so $f(x) = (x-1)(x-2)(x+1)$.



$$\int_{0}^{4} |x^3 - 2x^2 - x + 2|\,dx$$

$$= \int_{0}^{1} x^3 - 2x^2 - x + 2\,dx - \int_{1}^{2} x^3 - 2x^2 - x + 2\,dx + \int_{2}^{4} x^3 - 2x^2 - x + 2\,dx$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{1}{2}x^2 + 2x\right]_0^1 - \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{1}{2}x^2 + 2x\right]_1^2 + \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{1}{2}x^2 + 2x\right]_2^4$$

$$= \left(13/12 - 0\right) - \left(2/3 - 13/12\right) + \left(64/3 - 2/3\right)$$

$$= 133/6$$

$\sin x + \cos x = \sqrt{2}\sin(x + \pi/4)$

So $\displaystyle\int_{-\pi}^{\pi} |\sin x + \cos x|\, dx$

$\displaystyle = \int_{-\pi}^{\pi} |\sqrt{2}\sin(x + \pi/4)|\, dx$

$\displaystyle = \sqrt{2}\int_{-\pi}^{\pi} |\sin(x + \pi/4)|\, dx$

$\displaystyle = \sqrt{2}\int_{-\pi}^{\pi} |\sin x|\, dx$ because the interval has length 2π

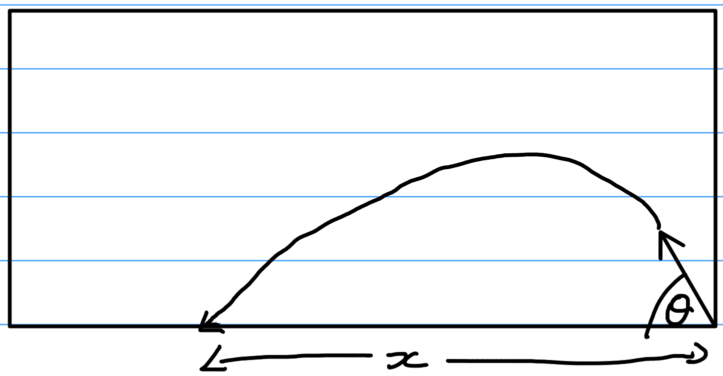
$\displaystyle = 2\sqrt{2}\int_{0}^{\pi} \sin x\, dx$ by symmetry

$= 2\sqrt{2}\left[-\cos x\right]_0^{\pi}$

$= 4\sqrt{2}$

STEP I 2000 Q9

The shell (relative to the carriage) experiences a fictitious force horizontally of magnitude ma.



$\leftarrow$		$\uparrow$	
s	x	s	0
u	$v\cos\theta$	u	$v\sin\theta$
v	$\times$	v	$\times$
a	a	a	$-g$
t	t	t	t

$x = vt\cos\theta + \frac{1}{2}at^2$

$0 = vt\sin\theta - \frac{1}{2}gt^2$

$t \neq 0 \Rightarrow v\sin\theta = \frac{1}{2}gt$

$\Rightarrow t = \frac{2v\sin\theta}{g}$

$$\Rightarrow x = \frac{2v^2\sin\theta\cos\theta}{g} + \frac{1}{2}a \cdot \frac{4v^2\sin^2\theta}{g^2}$$

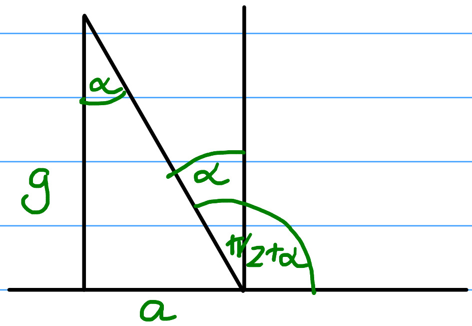
$$= \frac{v^2}{g}\sin 2\theta + \frac{av^2}{g^2}(1-\cos 2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = \cancel{\frac{2v^2}{g}}\cos 2\theta + \cancel{\frac{2av^2}{g^2}}\sin 2\theta = 0$$

$$\Rightarrow \cos 2\theta + \frac{a}{g}\sin 2\theta = 0$$

$\Rightarrow \tan 2\theta = -g/a$

If $a \ll g$, then $\tan 2\theta \ll -1 \Rightarrow 2\theta = \pi/2 + \alpha$ where $0 < \alpha \ll 1$. So,



$\Rightarrow \tan\alpha = \frac{a}{g}$ but $0 < \alpha \ll 1$, so $\tan\alpha \approx \frac{a}{g}$ by the small angle approximations.

So $2\theta \approx \pi/2 + a/g$

$\Rightarrow \theta \approx \pi/4 + a/2g$

$x = \frac{v^2}{g}\left(\sin 2\theta + \frac{a}{g}(1 - \cos 2\theta)\right)$

$= \frac{v^2}{g}\left(\sin\frac{\pi}{2}\cos\frac{a}{g} + \cos\frac{\pi}{2}\sin\frac{a}{g} + \frac{a}{g}\left(1 - \left(\cos\pi/2 \cos a/g - \sin\pi/2 \sin a/g\right)\right)\right)$

$\approx \frac{v^2}{g}\left(1 - \frac{1}{2}\left(\frac{a}{g}\right)^2 + 0 + \frac{a}{g}\left(1 - 0 + \frac{a}{g}\right)\right)$ (small angle approximations)

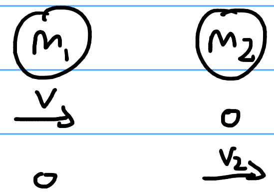
$= \frac{v^2}{g}\left(1 - \frac{a^2}{2g^2} + \frac{a}{g} + \frac{a^2}{g^2}\right)$

$= \frac{v^2}{g} + \frac{v^2 a}{g^2} + \frac{v^2 a^2}{2g^2}$ but $0 < a \ll g$, so $\frac{v^2a^2}{2g^2}$ is small compared to the other terms, so

$\approx \frac{v^2}{g} + \frac{v^2 a}{g^2}$, as required.

STEP I 2000 Q10

First Collision



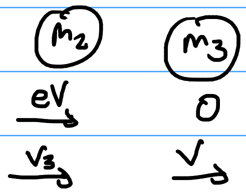
CoM: $m_1 V = m_2 V_2$

e : $eV = V_2$

$\Rightarrow m_1 V = e m_2 V$

$\Rightarrow m_1 = e m_2$

Second Collision



CoM: $m_2 e V = m_2 V_3 + m_3 V$ $\qquad \Rightarrow V_3 = \dfrac{V(em_2 - m_3)}{m_2} = \dfrac{V(m_1 - m_3)}{m_2}$ $\quad (*)$

e: $ee'V = V - V_3 \Rightarrow V_3 = V(1-ee')$

So $m_2 e\cancel{V} = m_2\cancel{V}(1-ee') + m_3\cancel{V}$

$\Rightarrow m_1 = m_2 - e'm_1 + m_3$ (as $em_2 = m_1$)

$\Rightarrow e' = \dfrac{m_2 + m_3 - m_1}{m_1}$

Note we require $0 \le e' \le 1$, and so $0 \le m_2 + m_3 - m_1 \le m_1$

$\Rightarrow m_1 \le m_2 + m_3 \le 2m_1$, as required.

Final energy $= \frac{1}{2} m_2 V_3^2 + \frac{1}{2} m_3 V^2$

$= \frac{1}{2} V^2 \left(m_2 \cdot \dfrac{(m_1 - m_3)^2}{m_2^2} + m_3 \right)$ (by $(*)$)

$$= \frac{1}{2}v^2\left(\frac{(m_1-m_3)^2}{m_2} + m_3\right)$$

And so maximising $E \Leftrightarrow$ minimising m_2 and vice versa.

We have $m_1 \leq m_2 + m_3 \leq 2m_1$

$$\Rightarrow m_1 - m_3 \leq m_2 \leq 2m_1 - m_3$$

but also $0 \leq e \leq 1 \Rightarrow 0 \leq \frac{m_1}{m_2} \leq 1 \Rightarrow m_2 \geq m_1$.

So, overall, $m_1 \leq m_2 \leq 2m_1 - m_3$

$$\text{So } E_{min} = \frac{1}{2}v^2\left(m_3 + \frac{(m_1-m_3)^2}{2m_1-m_3}\right)$$

$$= \frac{1}{2}v^2\left(\frac{2m_1m_3 - m_3^2 + m_1^2 - 2m_1m_3 + m_3^2}{2m_1 - m_3}\right)$$

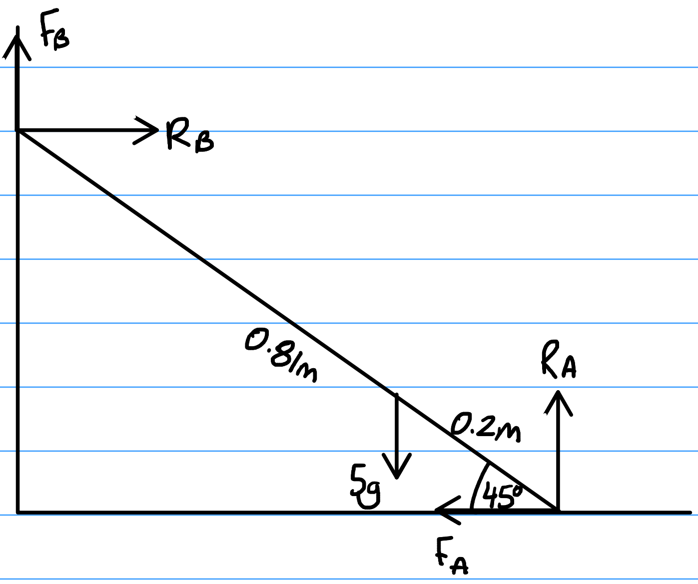
$$= \frac{1}{2}v^2\left(\frac{m_1^2}{2m_1-m_3}\right)$$

$$E_{max} = \frac{1}{2}v^2\left(m_3 + \frac{(m_1-m_3)^2}{m_1}\right)$$

$$= \frac{1}{2}v^2\left(\frac{m_1m_3 + m_1^2 - 2m_1m_3 + m_3^2}{m_1}\right)$$

$$= \frac{1}{2}v^2\left(\frac{m_1^2 - m_1m_3 + m_3^2}{m_1}\right)$$

11)



Resolving $\updownarrow$: $R_A + F_B = 5g$ ①

$\leftrightarrow$: $R_B = F_A$ ②

$M(A)$: $0.21 \times 5g = 0.81(F_B + R_B)$ ③ } note the cos45° & sin45° cancel out

$M(B)$: $0.6 \times 5g + 0.81F_A = 0.81R_A$ ④

Suppose friction is limiting at B, then $R_B = F_B$

Then ② $\Rightarrow F_A = R_B = F_B$

Then ③ gives $\frac{21}{20}g = \frac{81}{100}(2F_B)$

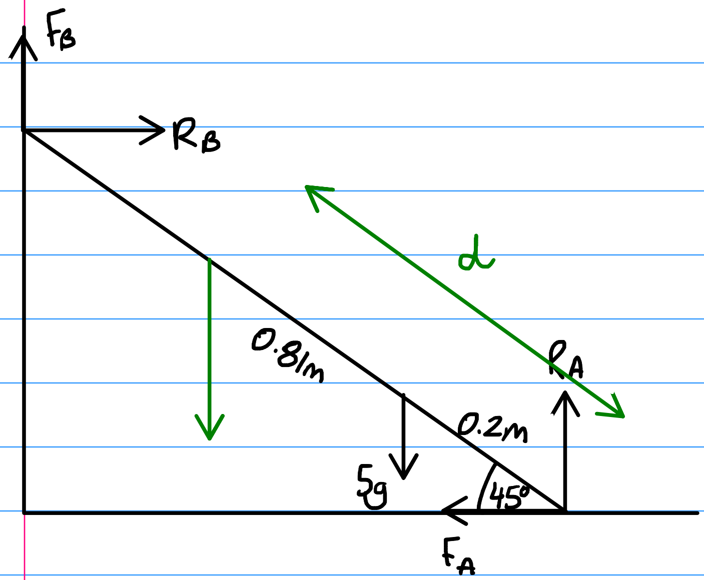
$\Rightarrow F_B = \frac{35}{54}g$

Then ① gives $R_A = 5g - F_B$

$= \frac{235}{54}g$

But if friction is limiting at A, then $\frac{35}{54}g = F_A = \frac{1}{5}R_A = \frac{1}{5} \times \frac{235}{54}g = \frac{47}{54}g$ ※.

So friction cannot be limiting at both A and B.



Resolving $\updownarrow$: $R_A + F_B = 10g$ ①

$\leftrightarrow$: $R_B = F_A$ ②

$M(A)$: $0.21 \times 5g + d \times 5g = 0.81(F_B + R_B)$ ③

$M(B)$: $0.6 \times 5g + 0.81F_A + (0.81 - d) \times 5g = 0.81R_A$ ④

Since friction is limiting at A and B, we have $R_B = F_B$ and $R_A = 5 \times F_A$. So ② then gives

$$F_B = R_B = F_A$$
$$R_A = 5F_A$$

Then ① becomes $6F_A = 10g \Rightarrow F_A = \frac{5}{3}g$

Then ③ becomes $5\cancel{g}(d + 0.21) = 0.81 \times 2 \times \frac{5}{3}\cancel{g}$

$\Rightarrow d + 0.21 = 0.54$

$\Rightarrow d = \underline{0.33m}$

STEP I 2000 Q12

(i) $P(\text{succeed on } n^{th} \text{ attempt}) = \underbrace{\frac{\cancel{k-1}}{k} \times \frac{\cancel{k-2}}{\cancel{k-1}} \times \cdots \times \frac{\cancel{k-(n-1)}}{\cancel{k-(n-2)}}}_{n-1 \text{ failures}} \times \underbrace{\frac{1}{\cancel{k-(n-1)}}}_{\text{success}}$

$$= \frac{1}{k}$$

(ii) $$= \left(\frac{k-1}{k}\right)^{n-1}\left(\frac{1}{k}\right)$$

$$= \frac{(k-1)^{n-1}}{k^n}$$

(iii) $$= \frac{k-1}{k} \times \frac{k-2}{k-1} \times \frac{k-2}{k-1} \times \cdots \times \frac{k-2}{k-1} \times \frac{1}{k-1}$$

$$= \frac{(k-1)(k-2)^{n-2}}{k(k-1)^{n-1}}$$

$$= \left(\frac{k-2}{k-1}\right)^{n-2} \times \frac{1}{k} \quad \text{for } n>1$$

or $\frac{1}{k}$ for $n=1$

STEP I 2000 Q13

(i) $\binom{4}{2} \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{4}\right)^2$

$= \frac{6}{256}$

$= \frac{3}{128}$

(ii) $P(F=AB \mid M=AB, C=AA) = \dfrac{P(F=AB, M=AB, C=AA)}{P(M=AB, F=AA)}$

$= \dfrac{P(F=AB, M=AB, C=AA)}{P(F=AB, M=AB, C=AA) + P(F=AA, M=AB, C=AA)}$

$= \dfrac{0.18 \times \cancel{0.18} \times \frac{1}{4}}{0.18 \times \cancel{0.18} \times \frac{1}{4} + 0.81 \times \cancel{0.18} \times \frac{1}{2}}$

$= \dfrac{0.09}{0.09 + 0.81}$

$= \frac{9}{90}$

$= \frac{1}{10}$

STEP I 2000 Q14

$X \sim U[-1,1]$, so $f(x) = \frac{1}{2}$ for $-1 \leq x \leq 1$.

$$EX^2 = \int_{-1}^{1} \tfrac{1}{2}x^2\,dx = \left[\tfrac{1}{6}x^3\right]_{-1}^{1}$$
$$= 1/3$$

$$EX^4 = \int_{-1}^{1} \tfrac{1}{2}x^4\,dx = \left[\tfrac{1}{10}x^5\right]_{-1}^{1}$$
$$= \frac{1}{5}$$

So $\mathrm{Var}X^2 = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}$

$$EZ^2 = E(Y-X)^2$$
$$= E(Y^2 - 2XY + X^2)$$
$$= EY^2 - 2EXEY + EX^2$$
$$= 1/3 - 0 + 1/3$$
$$= 2/3$$

$$EZ^4 = E(Y-X)^4$$
$$= E(Y^4 - 4Y^3X + 6Y^2X^2 - 4YX^3 + X^4)$$
$$= EY^4 - 4EY^3EX + 6EY^2EX^2 - 4EYEX^3 + EX^4$$
$$= \frac{1}{5} - 0 + 6 \times \frac{1}{3} \times \frac{1}{3} - 0 + \frac{1}{5}$$
$$= 16/15$$

So $\mathrm{Var}Z^4 = \frac{16}{15} - \left(\frac{2}{9}\right)^2 = \frac{28}{45} = 7 \times \frac{4}{45} = 7\mathrm{Var}X^2$