STEP III 1999 Q1

$x^3 - px^2 + qx - r = 0$

(i) We have roots a/k, a, ak.

So, $r = (a/k)(a)(ak) = a^3$

$q = (a/k)a + (a/k)ak + a(ak)$

$= a^2(1/k + 1 + k)$

$p = a/k + a + ak$

$= a(1/k + 1 + k)$

So $q^3 - rp^3 = a^6(1/k + 1 + k)^3 - a^3 \cdot a^3(1/k + 1 + k)^3$

$= 0$

Further, $\frac{q}{p} = \frac{a^2(1/k + 1 + k)}{a(1/k + 1 + k)} = a$ which is a root.

(ii) We have $x^3 - px^2 + qx - q^3/p^3 = 0$

So $f(q/p) = \frac{q^3}{p^3} - p \cdot \frac{q^2}{p^2} + q \cdot \frac{q}{p} - \frac{q^3}{p^3}$

$= 0$, so $x = q/p$ is a root.

Suppose the roots are α, β, γ, with $\beta = q/p$.

Then $\alpha\beta\gamma = q^3/p^3$

$\Rightarrow \frac{q}{p}\alpha\gamma = q^3/p^3$

$\Rightarrow \alpha\gamma = q^2/p^2$, as required.

Now suppose $\gamma = k^2 \times \alpha$ for some k.

Then $q^2/p^2 = \alpha\gamma$

$= \alpha \times k^2\alpha$

$= k^2\alpha^2$

So, $k = \frac{q}{p\alpha}$

But then $k\alpha = \frac{q\alpha}{p\alpha} = q/p = \beta$. So, with $k = \frac{q}{p\alpha}$, we have roots α, $k\alpha$, $k^2\alpha$, which is in geometric progression.

(iii) We have $p = \alpha + \beta + \gamma$
$q = \alpha\beta + \beta\gamma + \gamma\alpha$
$r = \alpha\beta\gamma$

If the roots are in arithmetic progression, then $\beta = \alpha + d$, $\gamma = \beta + d$ for some d.
So let $\alpha = \beta - d$, $\gamma = \beta + d$. Then

$$p = \beta - d + \beta + \beta + d$$
$$= 3\beta \qquad (1)$$

$$q = (\beta - d)(\beta + d) + \beta(\beta - d) + \beta(\beta + d)$$
$$= \beta^2 - d^2 + \beta^2 - d\beta + \beta^2 + d\beta$$
$$= 3\beta^2 - d^2 \qquad (2)$$

$$r = (\beta - d)\beta(\beta + d)$$
$$= \beta^3 - \beta d^2 \qquad (3)$$

$(2) \Rightarrow d^2 = 3\beta^2 - q$

Substituting into (3), $r = \beta^3 - \beta(3\beta^2 - q)$
$$= -2\beta^3 + q\beta \qquad (4)$$

But (1) gives $\beta = p/3$, so $r = -2(p/3)^3 + q(p/3)$
$$= -2p^3/27 + qp/3$$

$$\Rightarrow 27r = -2p^3 + 9qp$$
$$\Rightarrow 2p^3 - 9qp + 27r = 0$$

The argument works backwards too, so this is necessary and sufficient.

STEP III 1999 Q2

(i) $f(x) = (1+x^2)e^x$

$f'(x) = e^x(1+x^2+2x)$

$= (x+1)^2 e^x$

≥ 0 for all $x \in \mathbb{R}$, and $=0$ for $x=-1$.

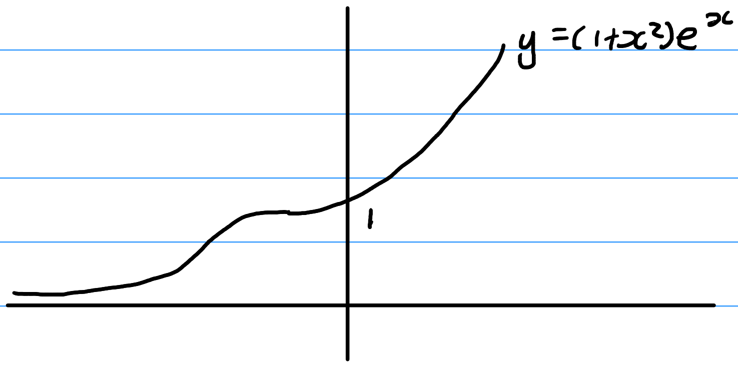
So f is increasing with a point of inflection at $x=-1$.

Further, $f(0)=1$

$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = 0$

So,



Clearly, from the graph, the equation $(1+x^2)e^x$ has exactly one solution for $k>0$ and no solutions for $k \leq 0$.

(ii) Let $f(x) = (e^x - 1) - k \arctan x$

Differentiating, $f'(x) = e^x - \frac{k}{1+x^2}$

So $f'(x) = 0 \implies (1+x^2)e^x = k$

By (i), this has exactly one solution, and for $0<k<1$ this solution is for $x<0$.

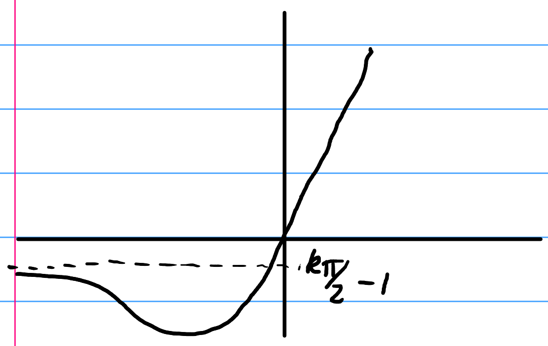
So f has exactly one stationary point, and this point has $x<0$.

$\lim_{x\to\infty} f(x) = \infty$

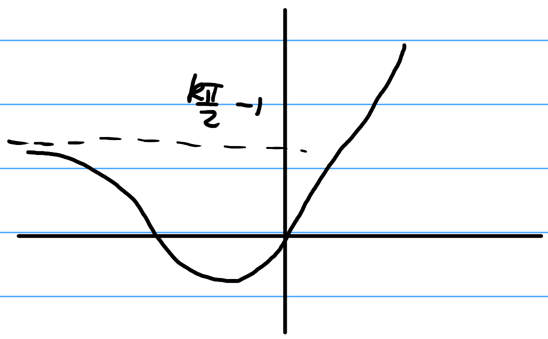
$f(0) = 0$

$$\lim_{x\to-\infty} f(x) = 1-k\cdot\left(-\tfrac{\pi}{2}\right)$$
$$= \frac{k\pi}{2} - 1$$

For $0<k\leq \frac{2}{\pi}$ this is ≤ 0, for $\frac{2}{\pi}<k<1$ this is >0. So,



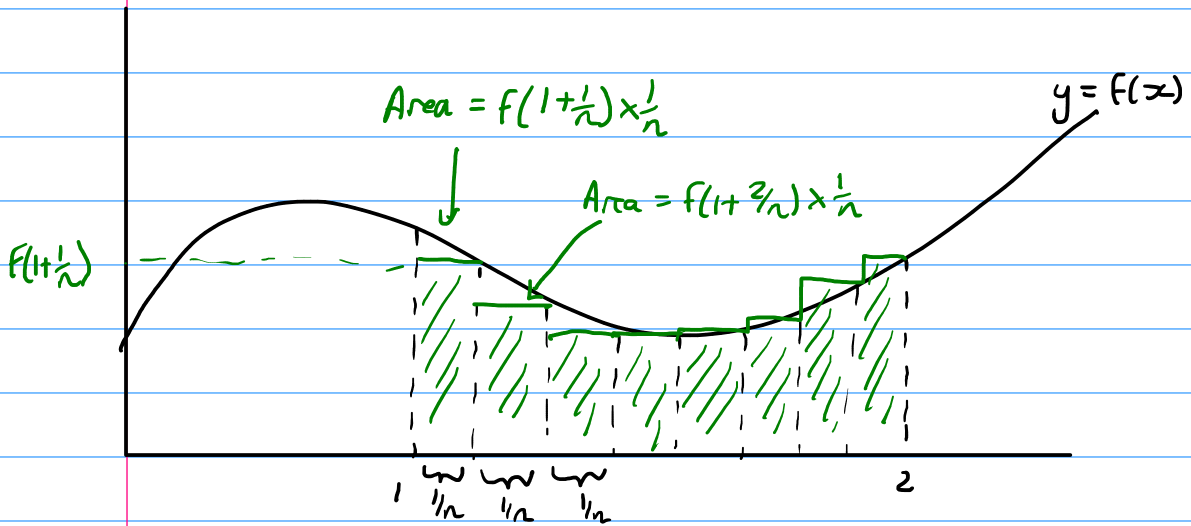
$0<k\leq \frac{2}{\pi}$



$\frac{2}{\pi}<k<1$

So $(e^x-1)-k\arctan x$ has exactly one solution for $0<k\leq \frac{2}{\pi}$, and exactly two solutions for $\frac{2}{\pi}<k<1$.

STEP III 1999 Q3



The green shaded area is $\sum_{m=1}^{n} f(1+\frac{m}{n})\times\frac{1}{n} = \frac{1}{n}\sum_{m=1}^{n} f(1+\frac{m}{n})$

In the limit as $n\to\infty$, this becomes the area under the curve, and so takes the value of the integral. So,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} f(1+m/n) = \int_1^2 f(x)\,dx$$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\text{---}+\frac{1}{n+n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\left(\frac{n}{n+1}+\frac{n}{n+2}+\text{---}+\frac{n}{n+n}\right)\right)$$

Now let $f(x)=\frac{1}{x}$, then $f(1+\frac{m}{n}) = f(\frac{n+m}{n}) = \frac{n}{n+m}$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\left(f(1+1/n)+f(1+2/n)+\text{---}+f(1+n/n)\right)\right)$$

$$= \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx$$

$= [\ln x]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2$

Now, want $\lim\limits_{n\to\infty} \frac{1}{n}\left(\frac{n^2}{n^2+1} + \frac{n^2}{n^2+4} + \text{—} + \frac{n^2}{n^2+n^2}\right)$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1+(\frac{1}{n})^2} + \frac{1}{1+(\frac{2}{n})^2} + \text{—} + \frac{1}{1+(\frac{n}{n})^2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \frac{1}{1+(m/n)^2}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \frac{1}{(1+m/n)^2 - 2(m/n)}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \frac{1}{(1+m/n)^2 - 2(1+m/n) + 2}$$

So with $f(x) = \frac{1}{x^2-2x+2}$, this becomes

$$\int_1^2 \frac{1}{x^2-2x+1}\,dx = \int_1^2 \frac{1}{(x-1)^2+1}\,dx$$

$$= [\arctan(x-1)]_1^2$$

$$= \arctan 1 - \arctan 0$$

$$= \pi/4$$

STEP III 1999 Q4

we have $V-E+F=2$.

The polygon has m-sided polygons, with n meeting at each vertex.

With F faces, we have $V = \frac{F\times m}{n}$

$E = \frac{F\times m}{2}$

So, $\frac{Fm}{n} - \frac{Fm}{2} + F = 2$

$\Rightarrow F(2m-nm+2n) = 4n$

So $F = \frac{4n}{2m-nm+2n}$

But $h = 4-(n-2)(m-2)$

$= 4-(nm-2m-2n+4)$

$= 2m-nm+2n$

So $F = \frac{4n}{h}$

Since $F, n > 0$ we must have $h>0$. So, $4-(n-2)(m-2)>0$

$\Rightarrow (n-2)(m-2) < 4$

Further, $n, m \geq 3$ (by simple geometry).

If $n=3$ $m=3,4,5$

$n=4$ $m=3$

$n=5$ $m=3$

$n \geq 6$ no solutions for m

so these are the only five regular polyhedra.

n	m	h	F	V	E	
3	3	3	4	4	6	tetrahedron
3	4	2	6	8	12	cube
3	5	1	12	20	30	dodecahedron
4	3	2	8	6	24	octahedron
5	3	1	20	12	30	icosahedron

STEP III 1999 Q5

We have $u_0=1, u_1=1, u_{n+1}=u_n+u_{n-1}$ for $n \geq 1$

$$u_{n+2}^2+u_{n-1}^2 = (u_{n+1}+u_n)^2+(u_{n+1}-u_n)^2$$
$$= 2(u_{n+1}^2+u_n^2), \text{ as required.}$$

Claim $u_{2n}=u_n^2+u_{n-1}^2$ AND $u_{2n+1}=u_{n+1}^2-u_{n-1}^2$

Proof by induction

$n=1$: $u_2=u_1^2+u_0^2$ $\qquad$ $u_3=u_2^2-u_0^2$

$2=1^2+1^2$ ✓ $\qquad$ $3=2^2-1^2$ ✓

Assume both statements are true for $n=k$. Then for $n=k+1$,

$$\begin{aligned} u_{2(k+1)} &= u_{2k+2} \\ &= u_{2k+1}+u_{2k} && \text{by definition} \\ &= u_{k+1}^2-u_{k-1}^2+u_k^2+u_{k-1}^2 && \text{by assumption} \\ &= u_{k+1}^2+u_k^2, \text{ as required.} \end{aligned}$$

$$\begin{aligned} u_{2(k+1)+1} &= u_{2k+3} \\ &= u_{2k+2}+u_{2k+1} && \text{by definition} \\ &= u_{k+1}^2+u_k^2+u_{k+1}^2-u_{k-1}^2 && \text{by assumption and previous result} \\ &= 2u_{k+1}^2+u_k^2-u_{k-1}^2 \\ &= (u_{k+2}^2+u_{k-1}^2-2u_k^2)+u_k^2-u_{k-1}^2 && \text{by the first result} \\ &= u_{k+2}^2-u_k^2, \text{ as required.} \end{aligned}$$

The results are true for $n=1$, and if true for $n=k$ then true for $n=k+1$. So true for all positive integers n.

STEP III 1999 Q6

$x^{2/n} + y^{2/n} = a^{2/n}$ $(*)$

$\Rightarrow (x^{1/n})^2 + (y^{1/n})^2 = (a^{1/n})^2$

Clearly $-a \le x, y \le a$. Set $x = a(\cos t)^n$

$y = a(\sin t)^n$

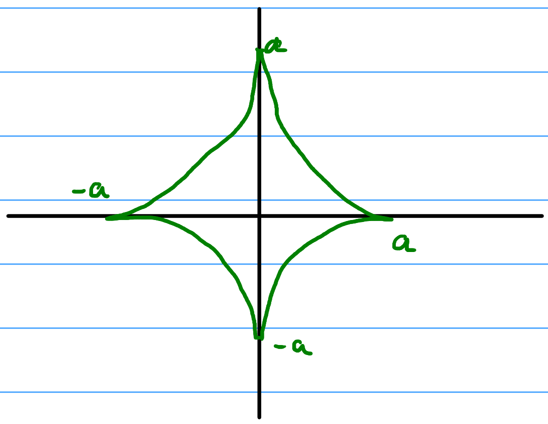
Then $(*)$ becomes

$$a^{2/n}(\cos^2 t + \sin^2 t) = a^{2/n}, \text{ which is trivially true.}$$

For $n=3$, $x = a\cos^3 t$

$y = a\sin^3 t$

Now, $(\cos t, \sin t)$ defines a circle. The cube will make values close to zero much closer to zero, but values near a will stay near a. There is four-fold rotational symmetry.



$$A = \frac{1}{2}\int_0^{2\pi} x\frac{dy}{dt} - y\frac{dx}{dt}\, dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} \cos^3 t \cdot 3\sin^2 t\cos t - \sin^3 t \cdot 3\cos^2 t(-\sin t)\, dt$$

$$= \frac{3a^2}{2}\int_0^{2\pi} \cos^4 t\sin^2 t + \sin^4 t\cos^2 t\, dt$$

$$= \frac{3a^2}{2}\int_0^{2\pi} \cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)\, dt$$

$$= \frac{3a^2}{2}\int_0^{2\pi} \cos^2 t \sin^2 t \, dt$$

$$= \frac{3a^2}{2}\int_0^{2\pi} \left(\frac{\sin 2t}{2}\right)^2 dt$$

$$= \frac{3a^2}{8}\int_0^{2\pi} \sin^2 2t \, dt$$

$$= \frac{3a^2}{8}\int_0^{2\pi} \frac{1}{2}(1+\cos 4t)\, dt$$

$$= \frac{3a^2}{16}\left[t + \frac{1}{4}\sin 4t\right]_0^{2\pi}$$

$$= \frac{3a^2\pi}{8}$$

STEP III 1999 7

$$(x*y)*z = (x+y+axy) + z + a(x+y+axy)z$$
$$= x+y+z + a(xy+yz+zx) + a^2xyz$$
$$x*(y*z) = x+(y+z+ayz) + ax(y+z+ayz)$$
$$= x+y+z + a(xy+yz+zx) + a^2xyz$$

So $*$ is associative.

We now wish to show that $(G,*)$ is a group

Associativity has been proved, and closure over the real numbers is clear.

For identity, need $x+y+axy=y$

$$\Rightarrow x+axy=0$$
$$\Rightarrow x(1+ay)=0$$
$$\Rightarrow x=0$$

So $x=0$ is the identity.

Inverse, need $x+y+axy=0$

$$\Rightarrow x = y(-1-ax)$$
$$\Rightarrow y = \frac{-x}{1+ax} \text{ is the inverse.}$$

But $x = -1/a$ has no inverse, so is not in the group. So $G = \mathbb{R}\backslash\{-1/a\}$

To be a subgroup of order 2, we need a self inverse element. So,

$$x+x+ax^2=0 \quad (x\neq 0)$$
$$\Rightarrow 2+ax=0$$
$$\Rightarrow x = -2/a$$

So, $\{0, -2/a\}$ is a subgroup of order 2.

STEP III 1999 Q8

At $x=0$, $\frac{dy}{dx}=1$ and $y=0$

$\frac{d^2y}{dx^2}+n^2y=0$ for $2(n-1)\pi < x < 2n\pi$, and y and $\frac{dy}{dx}$ are continuous at $x=2n\pi$

(i) For $0 \le x \le 2\pi$, $n=1$, so $\frac{d^2y}{dx^2}+y=0$

$\Rightarrow y = A\cos x + B\sin x$

$y(0)=0 \Rightarrow A=0$, $y'(0)=1 \Rightarrow B=1$

So $y=\sin x$

Note $y(2\pi)=0$, $y'(2\pi)=1$

(ii) $y''+4y=0 \Rightarrow y=A\cos 2x + B\sin 2x$

$A=0, B=1/2$, So $y=\frac{1}{2}\sin x$

Note $y(4\pi)=0$, $y'(4\pi)=0$

For $n \geq 1$, $y_n(x) = A\cos nx + B\sin nx$

Since $y(0)=0 \Rightarrow y(2\pi)=y(4\pi)=y(2m\pi)=0$, as $\sin(n\times 2k\pi)=0$ for $n,k\in\mathbb{Z}$.

Similarly, $y'(2m\pi)=1$, so $y_n(x)=\frac{1}{n}\sin nx$

So $y(x)=\frac{1}{n}\sin nx$ for $2(n-1)\pi \le x < 2\pi$

(iii) $\int_0^\infty y^2\,dx = \sum_{n=1}^{\infty}\int_{2(n-1)\pi}^{2n\pi} \frac{1}{n^2}(\sin nx)^2\,dx$

$= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{2\pi} \frac{1}{2}+\frac{1}{2}\cos(2nx)\,dx$

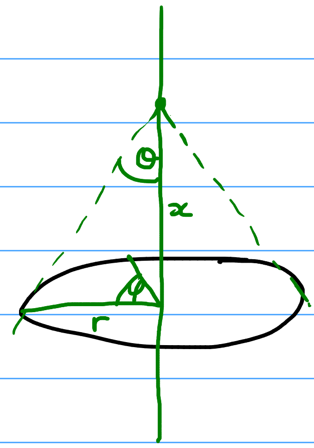
$= \sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\left[\frac{1}{2}x+\frac{1}{4n}\sin(2nx)\right]_0^{2\pi}$

$= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\pi\cdot\frac{1}{2}+0\right)$

$= \pi\sum_{n=1}^{\infty}\frac{1}{n^2}$, as required

STEP III 1999 Q9

For each angle φ, the 'left' and 'right' forces cancel out, so the only net force is towards the centre of the ring.

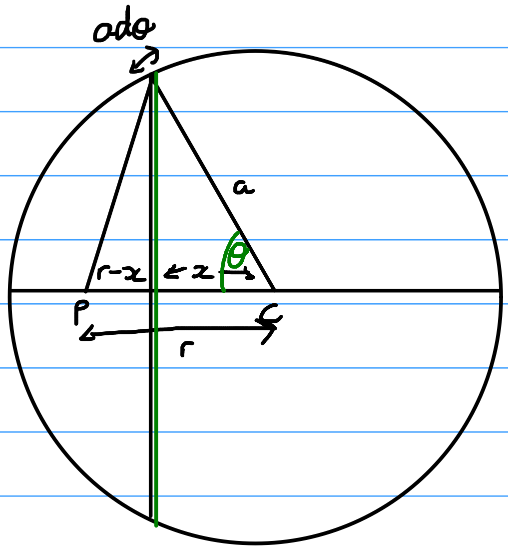


We have $F = \int_{\varphi=0}^{2\pi} \frac{G}{x^2+r^2} \cdot \cos\theta \cdot \frac{m}{2\pi}\, d\varphi$

$$= \frac{Gm\cos\theta}{x^2+r^2}$$

$$= \frac{Gm}{x^2+r^2} \cdot \frac{x}{\sqrt{x^2+r^2}}$$

$$= \frac{Gmx}{(x^2+r^2)^{3/2}}$$



Suppose the particle is located at P, at a distance of r from the centre of the shell.

The mass density of the shell is $\frac{M}{4\pi a^2}$, so the mass of the ring is

$$\frac{M}{4\pi a^2} \times 2\pi a\sin\theta \times a d\theta$$

density $\times 2\pi \times$ radius $\times$ width

$$= \tfrac{1}{2} M\sin\theta\, d\theta$$

So, the total force exerted on the particle is

distance to ring

mass of ring

$$F = \int_{\theta=0}^{\pi} \frac{G(r-x)\cdot \frac{1}{2} M\sin\theta\, d\theta}{(a^2-x^2+(r-x))^2}$$

radius of ring

distance to ring

But $\cos\theta = x/a$

$\Rightarrow \quad x = R\cos\theta$

$\Rightarrow \quad dx = -R\sin\theta \, d\theta$, and

So, $\quad F = -\int_{x=a}^{-a} \frac{Gm(r-x)}{2a(a^2-x^2+(r-x)^2)^{3/2}} \, dx$

$$= \frac{Gm}{2a} \int_{x=-a}^{a} \frac{(r-x)}{(a^2+r^2-2rx)^{3/2}} \, dx$$

Integrating by parts, $\quad u \quad r-x \qquad v' \quad \frac{1}{(a^2+r^2-2rx)^{3/2}}$

$\qquad u' \quad -1 \qquad V \quad \frac{1}{r(a^2+r^2-2rx)^{1/2}}$

$$= \left[\frac{r-x}{r(a^2+r^2-2rx)^{1/2}}\right]_{-a}^{a} + \int \frac{1}{r(a^2+r^2-2rx)^{1/2}} \, dx$$

$$= \left[\frac{r-x}{r(a^2+r^2-2rx)^{1/2}} - \frac{(a^2+r^2-2rx)^{1/2}}{r^2}\right]_{-a}^{a}$$

$$= \left[\frac{1}{(a^2+r^2-2rx)^{1/2}}\left(\frac{r-x}{r} - \frac{a^2+r^2-2rx}{r^2}\right)\right]_{-a}^{a}$$

$$= \left[\frac{1}{(a^2+r^2-2rx)^{1/2}}\left(\frac{r^2-x^2-a^2-r^2+2rx}{r^2}\right)\right]_{-a}^{a}$$

$$= \left[\frac{-1}{(a^2+r^2-2rx)^{1/2}} \cdot \left(\frac{x^2+a^2-2rx}{r^2}\right)\right]_{-a}^{a}$$

$$= \left(\frac{-1}{(a^2-2ra+r^2)^{1/2}} \cdot \frac{2a^2-2ar}{r^2}\right) - \left(\frac{-1}{(a^2+2ra+r^2)^{1/2}} \cdot \frac{2a^2+2ar}{r^2}\right)$$

$$= \frac{-2a(a-r)}{(a-r)r^2} + \frac{2a(a+r)}{(a+r)r^2}$$

$$= \frac{-2a}{r^2} + \frac{2a}{r^2}$$

$$= 0$$, as required.

STEP III 1999 Q10

The $(k+1)^{th}$ link falls through $\frac{k}{n}l$, so its speed when it collides with the table is $v^2 = 2g\frac{k}{n}l$. Hence the momentum is $\frac{m}{n}\sqrt{\frac{2gkl}{n}}$.

$$\text{Total impulse} = \sum_{k=0}^{n-1} \frac{m}{n}\sqrt{\frac{2gkl}{n}}$$

$$\approx m\sqrt{\frac{2gl}{n^3}} \int_1^n k^{1/2}\,dk$$

$$= m\sqrt{\frac{2gl}{n^3}}\left(\frac{2}{3}n^{3/2} - 2/3\right)$$

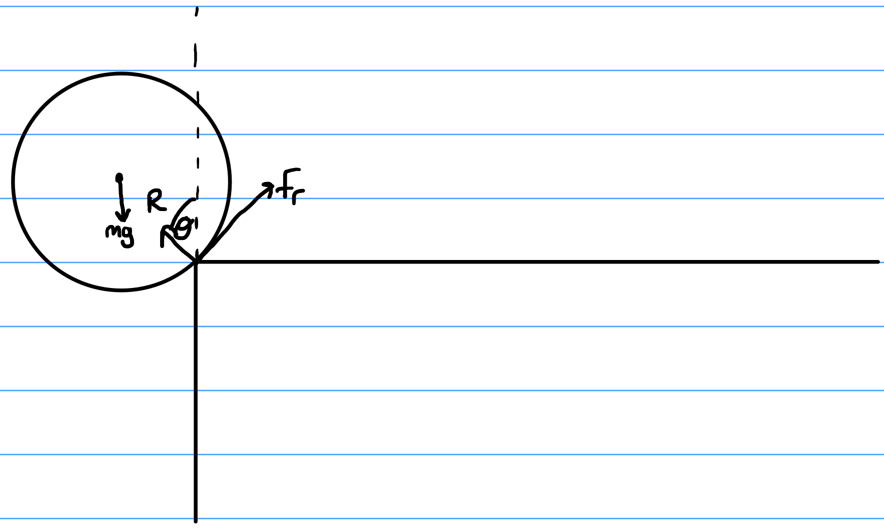
$$= \frac{2}{3}m\sqrt{2gl}\left(1 - \sqrt{\frac{1}{n^3}}\right)$$

In the limit $n \to \infty$, this is $\frac{2}{3}m\sqrt{2gl}$, as required.

STEP III 1999 Q11

By the parallel axis theorem, $I = I_{cm} + ma^2$

$$= ma^2 + ma^2$$
$$= 2ma^2$$



By conservation of energy, we have

$$mga(1-\cos\theta) = \frac{1}{2}\cdot 2ma^2\omega^2 - \frac{1}{2}\times 2mv^2 \quad (*)$$

Further, when the hoop is about to lose contact, $R=0$, so $mg\cos\theta = ma\omega^2$, so $(*)$ becomes

$$ga(1-\cos\theta) = ag\cos\theta - v^2$$
$$\Rightarrow \quad 2ag\cos\theta = ag + v^2$$
$$\Rightarrow \quad \cos\theta = \frac{1}{2} + \frac{v^2}{2ag}$$

STEP III 1999 Q12

$$P(X=j, Y=k) = e^{-1}\frac{(j+k)\lambda^{j+k}}{j!\,k!}$$

(i) $$P(X+Y=n) = \sum_{j=0}^{n} e^{-1}\frac{n\lambda^{n}}{j!(n-j)!}$$

$$= e^{-1}n\lambda^{n}\sum_{j=0}^{n}\frac{1}{j!(n-j)!}$$

$$= \frac{e^{-1}n\lambda^{n}}{n!}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}$$

$$= \frac{e^{-1}n\lambda^{n}}{n!}\sum_{j=0}^{n}\binom{n}{j}$$

$$= \frac{e^{-1}n\lambda^{n}2^{n}}{n!}$$

$$= \frac{e^{-1}(2\lambda)^{n}}{(n-1)!}$$

(ii) Noting that $P(X+Y=0)=0$, we must have

$$\sum_{n=1}^{\infty} P(X+Y=n) = 1$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{e^{-1}(2\lambda)^{n}}{(n-1)!} = 1$$

$$\Rightarrow 2\lambda e^{-1}\sum_{n=0}^{\infty}\frac{(2\lambda)^{n}}{n!} = 1$$

$$\Rightarrow 2\lambda e^{-1}e^{2\lambda} = 1$$

$$\Rightarrow 2\lambda e^{2\lambda-1} = 1$$

(iii) If $f(x) = 2xe^{2x-1}$, then $f'(x) = 2e^{2x-1}(1+2x) > 0$ for $x>0$. Hence the function is strictly increasing, so the equation has at most one solution. By inspection, $\lambda = 1/2$ solves the equation.

(iv) $E(2^{X+Y}) = \sum_{n=1}^{\infty} 2^n P(X+Y=n)$

$$= \sum_{n=1}^{\infty} 2^n \cdot \frac{e^{-1}(1)^n}{(n-1)!}$$

$$= 2e^{-1} \sum_{n=0}^{\infty} \frac{2^n}{n!}$$

$$= 2e^{-1} \times e^2$$

$$= 2e.$$

STEP III 1999 Q13

(i) $\int_0^1 Ax\,dx = 1$

$\Rightarrow \left[\frac{1}{2}Ax^2\right]_0^1 = 1$

$\Rightarrow \frac{1}{2}A = 1$

$\Rightarrow A = 2$

So $f(x) = 2x$.

$E(\text{size of portion}) = \int_0^1 2x^2\,dx$

$= \left[\frac{2}{3}x^3\right]_0^1$

$= 2/3$

So $E\,\text{currants} = 2/3 \times 4 = 8/3$

(ii) Let Y be the events taking a fraction y of the cake and getting four currants.

Then $f(y) = 2y \times y^4$

$= 2y^5$

$P(\text{all 4 currants}) = \int_0^1 2y^5\,dy = \left[\frac{1}{3}y^6\right]_0^1$

$= 1/3$

$P(\text{all 4 currants } \underline{\text{and}} \text{ take} > \frac{1}{2} \text{ cake}) = \int_{1/2}^1 2y^5\,dy = \left[\frac{1}{3}y^6\right]_{1/2}^1$

$= \frac{1}{3} - \frac{1}{192}$

$= 63/192$

So $P(>\frac{1}{2} \mid 4 \text{ currants}) = \dfrac{63/192}{1/3}$

$= 189/192$

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$$E \text{ winnings} = \sum_{i=1}^{n} q^{i-1} p i$$

$$= p \sum_{i=1}^{n} \frac{d}{dq}(q^i)$$

$$= p \frac{d}{dq} \sum_{i=1}^{n} q^i$$

$$= p \frac{d}{dq} \frac{q(1-q^n)}{1-q}$$

$$= p \, \frac{(1-(n+1)q^n)(1-q) + q(1-q^n)}{(1-q)^2}$$

$$= p^{-1}(1 - \cancel{q} - (n+1)q^n + (n+1)q^{n+1} + \cancel{q} - q^{n+1})$$

$$= p^{-1}(1 + nq^{n+1} - (n+1)q^n)$$, as required.

$$E\text{winning} = \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} p^{-1}(1 + nq^{n+1} - (n+1)q^n)$$

$$= p^{-1}\sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} + p^{-1}\sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda} q^{n+1}}{(n-1)!} - p^{-1}\sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda} q^n}{(n-1)!} - p^{-1}\sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda} q^n}{n!}$$

$$= p^{-1}\sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} + \lambda q^2 p^{-1} e^{-\lambda+\lambda q} \sum_{n=0}^{\infty} \frac{(\lambda q)^n e^{-\lambda q}}{n!} - p^{-1}\lambda q e^{-\lambda+\lambda q} \sum_{n=0}^{\infty} \underline{(\lambda q)^n e^{-\lambda q}}$$

$$- p^{-1} e^{-\lambda+\lambda q} \sum_{n=1}^{\infty} \frac{(\lambda q)^n e^{-\lambda q}}{n!}$$

$$= p^{-1}(1 - e^{-\lambda} + \lambda q^2 e^{-\lambda+\lambda q} - \lambda q e^{-\lambda+\lambda q} - e^{-\lambda+\lambda q}(1 - e^{-\lambda q}))$$

Noting that $e^{-\lambda+\lambda q} = e^{-(1-q)\lambda}$
$= e^{-p\lambda}$,

$$= p^{-1}(1 - \cancel{e^{-\lambda}} + \lambda q^2 e^{-p\lambda} - \lambda q e^{-p\lambda} - e^{-p\lambda} + \cancel{e^{-\lambda}})$$

$$= p^{-1}(1+\lambda(1-2p+p^2)e^{-p\lambda} - \lambda(1-p)e^{-p\lambda} - e^{-p\lambda})$$

$$= p^{-1}(1+\lambda e^{-p\lambda}(1-2p+p^2-1+p) - e^{-p\lambda})$$

$$= p^{-1}(1-e^{-p\lambda}) + \lambda p^{-1}e^{-p\lambda}(p^2-p)$$

$$= \frac{1}{p}(1-e^{-p\lambda}) - \lambda q e^{-p\lambda}$$, as required.