STEP II 1999 Q1

(i) $x! \approx \sqrt{2\pi}\, x^{x+1/2} e^{-x}$

$$\Rightarrow \log_{10} a_1 = \tfrac{1}{2}\log_{10}2\pi + (x+\tfrac{1}{2})\log_{10}x - x\log_{10}e$$
$$= \tfrac{1}{2}\log_{10}2\pi + 100x + 50 - x\log_{10}e$$
$$\approx 100x \quad \text{as } x>>1, \text{ and } 0<\log_{10}e<1$$

$$\Rightarrow \log_{10}\log_{10}a_1 = \log_{10}(100x)$$
$$= \log_{10}100 + \log_{10}x$$
$$= 2 + 100$$
$$= 102$$

(ii) $\log_{10}a_2 = y\log_{10}x$

$= 100y$

$\log_{10}\log_{10}a_2 = 2 + x$

$\log_{10}a_5 = xyz\log_{10}e$

$\log_{10}\log_{10}a_5 = \log_{10}x + \log_{10}y + \log_{10}z + \log_{10}\log_{10}e$

$\approx 100 + x + y$

$\log_{10}a_3 = x\log_{10}y$

$= x^2$

$\log_{10}\log_{10}a_3 = 2\log_{10}x$

$= 200$

$\log_{10}a_6 = \frac{1}{y}\log_{10}z$

$= \frac{1}{y}\cdot y$

$= 1$

$\log_{10}\log_{10}a_6 = 0$

$\log_{10}a_4 = x\log_{10}z$

$= xy$

$\log_{10}\log_{10}a_4 = \log_{10}x + \log_{10}y$

$= 100 + x$

$\log_{10}a_7 = \frac{z}{x}\log_{10}y$

$= z$

$\log_{10}\log_{10}a_7 = \log_{10}z$

$= y$

Using the fact that $\log a < \log b \Leftrightarrow a<b$ (as log is strictly increasing), and $1 << x << y << z$, we have

$$a_6 < a_1 < a_3 < a_2 < a_4 < a_7 < a_5$$

STEP II 1999 Q2

$nx^2+2x\sqrt{pn^2+q}+rn+s=0$

(i) $p=3,\ q=50,\ r=2,\ s=15$

$nx^2+2x\sqrt{3n^2+50}+2n+15=0$

$b^2-4ac<0 \Rightarrow 12n^2+200-8n^2-60n<0$

$\Rightarrow 4n^2-60n+200<0$

$\Rightarrow n^2-15n+50<0$

$\Rightarrow (n-10)(n-5)<0$

$\Rightarrow \quad 5<n<10$

$\Rightarrow n=6,7,8,9.$

(ii) $b^2-4ac=4pn^2+4q-4n(rn+s)$

$=4[n^2(p-r)-sn+q]$

Considering this as a quadratic in n, and as $p<r$ this is a negative quadratic. The discriminant of this quadratic is $s^2-4q(p-r)<0$ as $4q(p-r)>s^2$, the discriminant is less than zero. So this is a negative quadratic that doesn't cross the x-axis so is always negative. Hence the discriminant of the original quadratic is always negative and hence (*) has no real solutions.

(iii) Now the discriminant is $1-s+s^2/8 \geq 0$ (†)

The roots are $s=\dfrac{1\pm\sqrt{1-1/2}}{1/4}$

$=4\pm4\sqrt{1/2}$

$=4\pm2\sqrt{2}$

So (†) becomes $s\leq 4-2\sqrt{2}$ or $s\geq 4+2\sqrt{2}$, as required.

STEP II 1999 Q3

$$S_1(x) = e^{x^3}\frac{d}{dx}(e^{-x^3})$$
$$= e^{x^3}(-3x^2e^{-x^3})$$
$$= -3x^2$$

$$S_2(x) = e^{x^3}\frac{d}{dx}(-3x^2e^{-x^3})$$
$$= e^{x^3}(9x^4-6x)e^{-x^3}$$
$$= 9x^4-6x$$

$$S_3(x) = e^{x^3}(36x^3-6-27x^6+18x^3)e^{-x^3}$$
$$= -27x^6+54x^3-6$$

Claim $S_n(x)$ is a polynomial

Proof True for $n=1$, as above.

Assume true for $n=k$, so $\frac{d^k}{dx^k}(e^{-x^3}) = e^{-x^3}f_k(x)$ where f_k is a polynomial.

Then $S_{k+1}(x) = e^{-x^3}\frac{d}{dx}(e^{-x^3}f_k(x))$
$$= e^{-x^3}(f_k'(x) - 3x^2f_k(x))e^{-x^3}$$
$= f_k'(x) - 3x^2f_k(x)$, which is a polynomial.

True for $n=1$, and if true for $n=k$ then true for $n=k+1$, so true for all $n \in \mathbb{N}$.

The order of f_k increases by 2 each time, so the order of $S_n(x)$ is $2n$, and the coefficient of x^{2n} is $(-3)^n$, as it is multiplied by -3 each time.

Now $\frac{dS_n}{dx} = 0$ for some $x=a$. But from above, $S_{n+1}(x) = S_n'(x) - 3x^2S_n(x)$, and $S_n'(a)=0$, so $S_{n+1}(a) = -3a^2S_n(a)$. $3a^2>0$, so for both sides to be equal, exactly one of $S_{n+1}(a)$ and $S_n(a)$ are negative. So $S_n(a)S_{n+1}(a) \leq 0$.

STEP II 1999 Q4

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n}$$

Consider the coefficient of x^n. On the LHS, this is $\sum_{s=0}^{n}\binom{n}{s}\binom{n}{n-s} = \sum_{s=0}^{n}\binom{n}{s}^2$. On the RHS it's just $\binom{2n}{n}$. So,

$$\sum_{s=0}^{n}\binom{n}{s}^2 = \binom{2n}{n}.$$

(i)

$$(1-x)^n(1+x)^n \equiv (1-x^2)^{2n}$$

Consider the coefficient of x^n. On the LHS, this is $\sum_{=0}^{n}(-1)^n\binom{n}{s}\binom{n}{n-s} = \sum_{s=0}^{n}(-1)^n\binom{n}{s}^2$. On the RHS, this is $(-1)^{n/2}\binom{n}{n/2}$, so

$$\sum_{s=0}^{n}(-1)^n\binom{n}{s}^2 = (-1)^{n/2}\binom{n}{n/2}$$

(ii)

$$(1+x)^{n-1}(1+x)^n \equiv (1+x)^{2n-1}$$

Considering the coefficient of x^n,

$$\sum_{t=1}^{n}\binom{n}{t}\binom{n-1}{n-t} = \binom{2n-1}{n}$$

$$\Rightarrow \sum_{t=1}^{n}\binom{n}{t}\cdot\frac{(n-1)!}{(n-t)!(t-1)!} = \binom{2n-1}{n}$$

$$\Rightarrow \sum_{t=1}^{n}\binom{n}{t}\cdot\frac{t}{n}\binom{n}{t} = \binom{2n-1}{n}$$

$$\Rightarrow \sum_{t=1}^{n}2t\binom{n}{t}^2 = 2n\cdot\frac{(2n-1)!}{n!(n-1)!}$$

$$= 2n\cdot\frac{(2n)!}{n!n!}\cdot\frac{n}{2n}$$

$$= n\binom{2n}{n}, \text{ as required.}$$

STEP II 1999 Q5

$$5\cos x + 12\sin x = 7$$

$$\Rightarrow 12\sin x = 7 - 5\cos x$$

$$\Rightarrow 144\sin^2 x = (7 - 5\cos x)^2$$

$$\Rightarrow 144(1 - \cos^2 x) = 49 - 70\cos x + 25\cos^2 x$$

$$\Rightarrow 0 = 169\cos^2 x - 70\cos x - 95$$

$$\Longrightarrow \cos\alpha = \frac{70 \pm \sqrt{70^2 + 4 \times 169 \times 95}}{2 \times 169}$$

$$= \frac{70 \pm 2\sqrt{35^2 + 169 \times 95}}{2 \times 169}$$

$$= \frac{35 \pm \sqrt{5(7 \times 35 + 19 \times 169)}}{169}$$

$$= \frac{35 \pm \sqrt{5(245 + 3680 - 169)}}{169}$$

$$= \frac{35 \pm \sqrt{5 \times 3456}}{169}$$

$$= \frac{35 \pm \sqrt{5 \times 144 \times 24}}{169}$$

$$= \frac{35 \pm 12\sqrt{120}}{169}$$

, as required.

Now $\pi/2 < \alpha < \pi$, so $\cos\alpha < 0 \Rightarrow \cos\alpha = \frac{35 - 12\sqrt{120}}{169}$. We want to show $\pi/2 < \alpha < 3\pi/4$
$\Leftrightarrow 0 > \cos\alpha > -\sqrt{2}/2$.

So we must show $\frac{35 - 12\sqrt{120}}{169} > \frac{-\sqrt{2}}{2}$

$$\Leftrightarrow 70 - 24\sqrt{120} > -169\sqrt{2}$$
$$\Leftrightarrow 70 + 169\sqrt{2} > 24\sqrt{120}$$

$\sqrt{2} > 1.4$, so the LHS is $> 70 + 169 \times 1.4$
$$= 70 + 236.6$$
$$= 306.6$$
$$> 264$$
$$= 24 \times 11$$
$$= 24\sqrt{121}$$
$$> 24\sqrt{120}$$, as required.

So $\alpha < 3\pi/4$.

STEP II 1999 Q6

$$y = \frac{ax+b}{cx+d} \qquad \frac{dy}{dx} = \frac{a(cx+d)-c(ax+b)}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2}$$

$$\int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x+1}{x+3}\right) dx \qquad \text{set } u = \frac{x+1}{x+3}, \text{ so } \frac{du}{dx} = \frac{2}{(x+3)^2}$$

$$= \int_{1/3}^{1/2} \frac{1}{(x+3)^2} \ln u \quad \frac{(x+3)^2}{2} du$$

$$= \frac{1}{2}\int_{1/3}^{1/2} \ln u \, du$$

$$= \frac{1}{2}\left[u\ln u - u\right]_{1/3}^{1/2}$$

$$= \frac{1}{2}\left(\frac{1}{2}\ln 2 - \frac{1}{2} - \frac{1}{3}\ln\frac{1}{3} + \frac{1}{3}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{2}\ln 2 + \frac{1}{3}\ln 3 - \frac{1}{6}\right)$$

$$= \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12}$$

$$\int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{(x+1)(x+2)}{(x+3)^2}\right) dx$$

$$= \int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x+1}{x+3}\right) dx + \int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x+2}{x+3}\right) dx \qquad \text{set } u = \frac{x+2}{x+3}, \ \frac{du}{dx} = \frac{1}{(x+3)^2}$$

$$= \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12} + \int_{2/3}^{3/4} \ln u \, du$$

$$= \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12} + \left[u\ln u - u\right]_{2/3}^{3/4}$$

$$= \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12} + \left(\frac{3}{4}\ln\frac{3}{4} - \frac{3}{4} - \frac{2}{3}\ln\frac{2}{3} + \frac{2}{3}\right)$$

$$= \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12} + \left(\frac{3}{4}\ln 3 - \frac{3}{2}\ln 2 - \frac{3}{4} - \frac{2}{3}\ln 2 + \frac{2}{3}\ln 3 + \frac{2}{3}\right)$$

$$= \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12} + \left(\frac{17}{12}\ln 3 - \frac{13}{6}\ln 2 - \frac{1}{12}\right)$$

$$= \frac{19}{12}\ln 3 - \frac{29}{12}\ln 2 - \frac{1}{6}$$

$$\int_0^1 \frac{1}{(x+3)^2}\ln\left(\frac{x+1}{x+2}\right) dx$$

$$= \int_0^1 \frac{1}{(x+3)^2}\ln\left(\frac{(x+1)}{(x+3)} \cdot \frac{(x+3)}{(x+2)}\right) dx$$

$$= \int_0^1 \frac{1}{(x+3)^2}\ln\left(\frac{x+1}{x+3}\right) dx - \int_0^1 \frac{1}{(x+3)^2}\ln\left(\frac{x+2}{x+3}\right) dx$$

$$= \frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12} - \left(\frac{17}{12}\ln 3 - \frac{13}{6}\ln 2 - \frac{1}{12}\right)$$

$$= -\frac{5}{4}\ln 3 + \frac{23}{12}\ln 2$$

STEP II 1999 Q7

$$y = \frac{x}{\sqrt{x^2-2x+a}} = \frac{x}{\sqrt{(x-1)^2+a-1}}$$

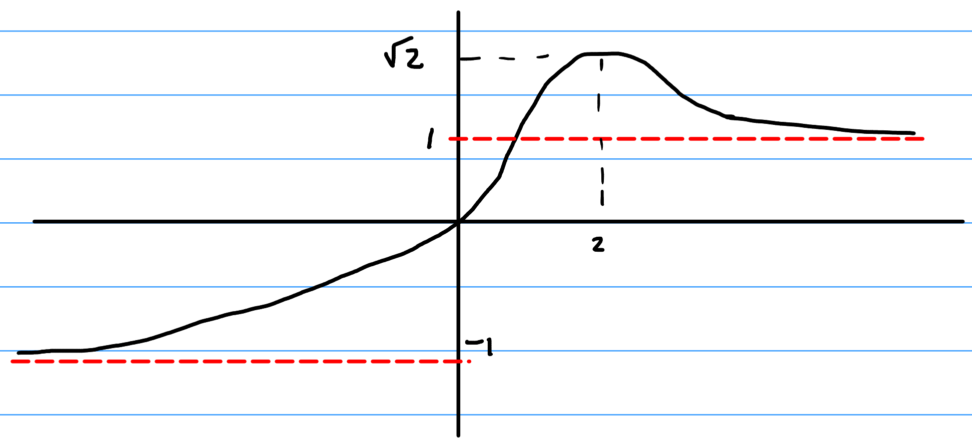
$a>1$ ensures the function is continuous.

$$\frac{dy}{dx} = \frac{\sqrt{x^2-2x+a} - x\cdot\frac{1}{2}\cdot(2x-2)\cdot\frac{1}{\sqrt{x^2-2x+a}}}{x^2-2x+a}$$

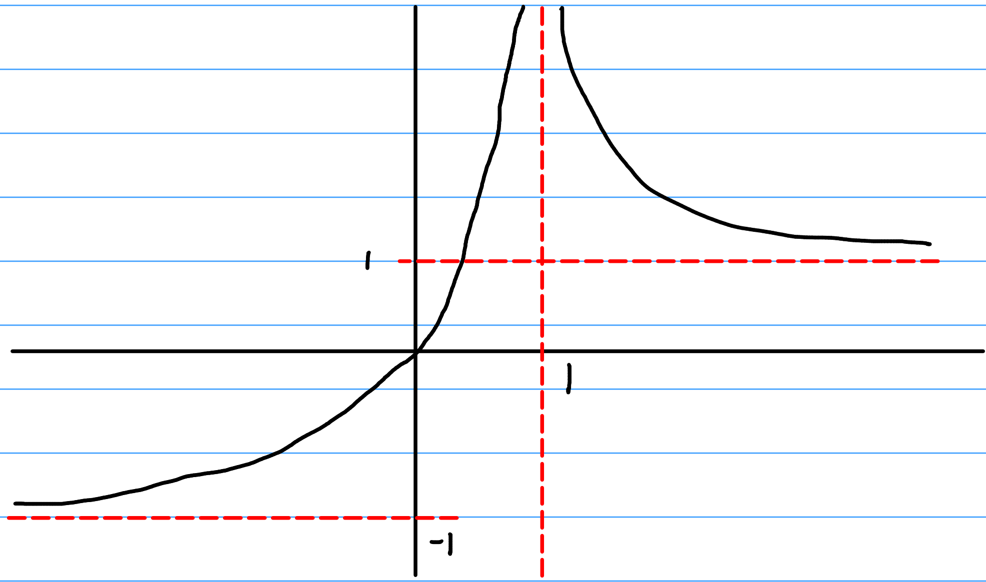
$$\frac{dy}{dx}=0 \Rightarrow x^2-2x+a-x(x-1)=0$$
$$\Rightarrow x^2-2x+a-x^2+x=0$$
$$\Rightarrow x=a$$

So, if $a>1$, exactly one stationary point at $x=a$.

For $a=2$, there is a SP at $x=2$.
As $x\to\infty$, $y\to 1$
As $x\to-\infty$, $y\to -1$
When $x=0$, $y=0$
When $x=2$, $y=2/\sqrt{2}=\sqrt{2}$



For $a=1$, asymptote at $x=1$. The other properties are the same as previously.



STEP II 1999 Q8

$$\sum_{k=0}^{n} 2\sin\tfrac{1}{2}\theta\sin k\theta = \sum_{k=0}^{n} \cos(k-\tfrac{1}{2})\theta - \cos(k+\tfrac{1}{2})\theta$$

$$= \cos(-\tfrac{1}{2}\theta) - \cos(n+\tfrac{1}{2})\theta$$

$$= \cos\tfrac{1}{2}\theta - \cos(n+\tfrac{1}{2})\theta$$

So $$\sum_{k=0}^{n} \sin k\theta = \frac{\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

(i) Setting $\theta = \pi/n$,

$$\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right) = \frac{\cos(\pi/2n) - \cos\left(\frac{n+1/2}{n}\right)\pi}{2\sin \pi/2n}$$

$$\approx \frac{1 - \frac{\pi^2}{8n^2} - \cos\pi}{2\cdot \pi/2n} \qquad \text{as } \frac{n+1/2}{n} \approx 1$$

$$= \frac{2n - \pi^2/8n}{\pi}$$

$$\approx \frac{2n}{\pi} \qquad \text{as } n \text{ is large}$$

(ii) Differentiating,

$$\sum_{k=0}^{n} k\cos k\theta = \frac{2\sin\frac{1}{2}\theta\left(-\frac{1}{2}\sin\frac{1}{2}\theta + (n+\frac{1}{2})\sin(n+\frac{1}{2})\theta\right) - \cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta\right)}{4\sin^2\frac{1}{2}\theta}$$

$$= \frac{-\sin^2\frac{1}{2}\theta + 2(n+\frac{1}{2})\sin\frac{1}{2}\theta\sin(n+\frac{1}{2})\theta - \cos^2\frac{1}{2}\theta + \cos\frac{1}{2}\theta\cos(n+\frac{1}{2})\theta}{4\sin^2\frac{1}{2}\theta}$$

$$= \frac{-1 + (2n+1)\sin\frac{1}{2}\theta\sin(n+\frac{1}{2})\theta + \cos\frac{1}{2}\theta\cos(n+\frac{1}{2})\theta}{4\sin^2\frac{1}{2}\theta}$$

But also $\displaystyle\sum_{k=0}^{n} k\cos k\theta = \sum_{k=0}^{n} k - 2k\sin^2\frac{k\theta}{2}$

And so $\displaystyle\sum_{k=0}^{n} k\sin^2\frac{k\theta}{2} = \frac{1}{2}\sum_{k=0}^{n} k + \frac{1-(2n+1)\sin\frac{1}{2}\theta\sin(n+\frac{1}{2})\theta - \cos\frac{1}{2}\theta\cos(n+\frac{1}{2})\theta}{4\sin^2\frac{1}{2}\theta}$

Setting $\theta = \pi/n$,

$$\sum_{k=0}^{n} k\sin^2\frac{k\pi}{2n} = \frac{1}{4}n(n+1) + \frac{1-(2n+1)\sin\frac{\pi}{2n}\sin\left(\frac{n+1/2}{n}\right)\pi - \cos\frac{\pi}{2n}\cos\left(\frac{n+1/2}{n}\right)\pi}{8\sin^2\pi/2n}$$

n is large, so using small angle approximations, and $\frac{n+1/2}{n} \approx 1$,

$$= \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1-(2n+1)\cdot\frac{\pi}{2n}\cdot\overset{=0}{\sin\pi} - \left(1-\frac{\pi^2}{8n^2}\right)\overset{=-1}{\cos\pi}}{8\cdot\pi^2/4n^2}$$

$$= \frac{1}{4}n^2 + \frac{1}{4}n + \frac{n^2}{2\pi^2}\left(1+1-\pi^2/8n^2\right)$$

$$= \frac{1}{4}n^2 + \frac{1}{4}n + \frac{n^2}{\pi^2} + \frac{1}{16}$$ but $n^2 >> n, 1$, so

$$\approx \frac{1}{4}n^2 + \frac{1}{\pi^2}n^2$$

$$= \left(\frac{1}{4} + \frac{1}{\pi^2}\right)n^2$$, as required.

STEP II 1999 Q9

(i) $\frac{dv}{dt} = -k_1 x^3$

$\Rightarrow \frac{dx}{dt}\cdot\frac{dv}{dx} = -k_1 x^3$

$\Rightarrow v\frac{dv}{dx} = -k_1 x^3$

$\Rightarrow \frac{1}{2}v^2 = \frac{-k_1}{4}x^4 + c$

$v(0) = u \Rightarrow c = \frac{1}{2}u^2$

So $v^2 = u^2 - \frac{k_1}{2}x^4$ $\qquad (*)$

(ii) We have $u^2 < a^2 = \frac{k_1}{2}R^4$

But by $(*)$ we must have $u^2 \geqslant \frac{k_1}{2}x^4$, so $\frac{k_1}{2}x^4 \leqslant u^2 < \frac{k_1}{2}R^4$

$\Rightarrow x^4 < R^4$

$\Rightarrow x < R \quad (\text{as } k_1, x, R > 0)$

(iii) Now $u > a$, so the fragment goes past R. When $x = R$, $u_R^2 = u^2 - \frac{k_1}{2}R^4$

After meeting R, $v\frac{dv}{dx} = -k_2 x^{-4}$

$\Rightarrow \frac{1}{2}v^2 = \frac{k_2}{3}x^{-3} + c$

$v(R) = u_R$, so $\frac{1}{2}u_R^2 = \frac{k_2}{3}R^{-3} + c$

$\Rightarrow c = \frac{1}{2}u_R^2 - \frac{k_2}{3}R^{-3}$

So, $\frac{1}{2}v^2 = \frac{k_2}{3}x^{-3} + \frac{1}{2}u^2 - \frac{k_1}{4}R^4 - \frac{k_2}{3}R^{-3}$

$\Rightarrow v^2 = u^2 - \frac{k_1}{2}R^4 - \frac{2k_2}{3}R^{-3} + \frac{2k_2}{3}x^{-3}$

To be always moving away from G, require $v^2 > 0$ (as $v(0) > 0$)

As $x \to \infty$, $x^{-3} \to 0$, and so need

$u^2 - \frac{k_1}{2}R^4 - \frac{2k_2}{3}R^{-3} \geqslant 0$

$\Rightarrow 6u^2 \geqslant 3k_1R^4 + 4k_2R^{-3}$

So $u \geqslant b$, as required.

(iv) Maximum distance is where $v = 0$

$$\Rightarrow u^2 - \frac{k_1}{2}R_4 - \frac{2k_2}{3}R^{-3} + \frac{2}{3}k_2 x^{-3} = 0$$

$$\Rightarrow u^2 - b^2 + \frac{2}{3}k_2 x^{-3} = 0$$

$$\Rightarrow k_2 x^{-3} = \frac{3(b^2 - u^2)}{2}$$

$$\Rightarrow x = \sqrt[3]{\frac{2k_2}{3(b^2 - u^2)}}$$

STEP II 1999 Q10

Initial $\longrightarrow V$

Final $\longrightarrow v_1$ $\longrightarrow v_2$

m qm

CoM: $mV = mv_1 + qmv_2 \Rightarrow V = v_1 + qv_2$ (1)

e: $eV = v_2 - v_1$ (2)

(1)+(2) $\Rightarrow V(1+e) = v_2(1+q)$

$\Rightarrow v_2 = \left(\frac{1+e}{1+q}\right)V$

$$\begin{aligned} v_1 &= v_2 - eV \\ &= V\left(\frac{1+e}{1+q} - e\right) \\ &= V\left(\frac{1+e-e-eq}{1+q}\right) \\ &= \left(\frac{1-eq}{1+q}\right)V \text{, as required.} \end{aligned}$$

All collisions have the same mass ratio, so the final velocities are in the same ratio.

For N-1 impacts, require $V_k \geq V_{k-1}$ (otherwise $k-1$ 'catches up' with k)

So need $\left(\frac{1+e}{1+q}\right)^{k-1}\left(\frac{1-eq}{1+q}\right) \geq \left(\frac{1+e}{1+q}\right)^{k-2}\left(\frac{1-eq}{1+q}\right)$ Note $q<e(<1)$ means $1-eq$ is positive.

$\Leftrightarrow \frac{1+e}{1+q} \geq 1$

$\Leftrightarrow 1+e \geq 1+q$

$\Leftrightarrow e \geq q$

In the k^{th} collision, the initial KE is $\frac{1}{2}e^{k-1}mV^2$

final KE is $\frac{1}{2}e^{k-1}m(1-e)^2V^2 + \frac{1}{2}e^k mV^2$

So KE loss is $\frac{1}{2}mV^2e^{k-1}(1-(1-e)^2-e)$

$$= \frac{1}{2}mV^2e^{k-1}(1-1+2e-e^2-e)$$

$$= \frac{1}{2}mV^2e^{k-1}(e-e^2)$$

$$= \frac{1}{2}mV^2e^{k}(1-e)$$

So total KE loss is $\sum_{k=1}^{N-1} \frac{1}{2}mV^2e^{k}(1-e)$

$$= \frac{1}{2}mV^2(1-e)\cdot\frac{e(1-e^{N-1})}{1-e}$$

$$= \frac{1}{2}me(1-e^{N-1})V^2$$, as required

STEP II 1999 Q11

For the shell, vertically $s = ut + \frac{1}{2}at^2$

$$0 = v\sin\alpha\, t - \frac{1}{2}gt^2$$

$$\Rightarrow t = \frac{2v\sin\alpha}{g} \quad (\text{or } 0)$$

So range $= \frac{2v\sin\alpha}{g} \times v\cos\alpha$

$$= \frac{v^2 \sin 2\alpha}{g}$$

This has a maximum of v^2/g when $\alpha = 45°$. So the shell cannot reach the target if $v^2/g < R \Leftrightarrow v^2 < gR$.

Now, we have $R = \frac{2v^2}{g}\sin\alpha\cos\alpha$ $(*)$ from previously. Also, the time of flight is $\frac{2v\sin\alpha}{g}$, and so $\frac{2v\sin\alpha}{g}\omega = \beta \Rightarrow \sin\alpha = \frac{g\beta}{2\omega v}$ and $\cos\alpha = \sqrt{1 - \frac{g^2\beta^2}{4\omega^2 v^2}}$.

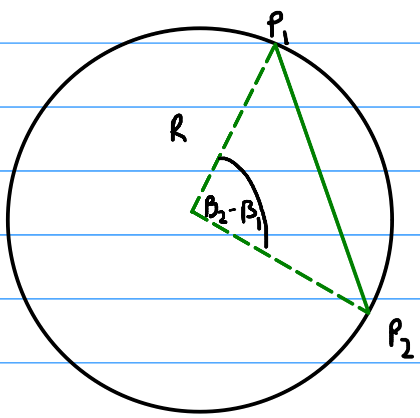
Substituting into $(*)$, $R = \frac{2v^2}{g} \cdot \frac{g\beta}{2\omega v} \cdot \sqrt{1 - \frac{g^2\beta^2}{4\omega^2 v^2}}$

$$\Rightarrow \frac{R\omega}{\beta v} = \sqrt{1 - \frac{g^2\beta^2}{4\omega^2 v^2}}$$

$$\Rightarrow \frac{R^2\omega^2}{\beta^2 v^2} = 1 - \frac{g^2\beta^2}{4\omega^2 v^2}$$

$$\Rightarrow 4R^2\omega^4 = 4\omega^2\beta^2 v^2 - g^2\beta^4$$

$$\Rightarrow g^2\beta^4 - 4\omega^2\beta^2 v^2 + 4R^2\omega^4 = 0$$, as required.



wlog $\beta_2 - \beta_1$.

So the distance is $2 \times R \times \sin\frac{1}{2}(\beta_2 - \beta_1)$

But $\beta_1^2 + \beta_2^2 = 4\omega^2 v^2/g^2$

$\beta_1^2 \beta_2^2 = 4R^2\omega^4/g^2$

So $(\beta_2 - \beta_1)^2 = \beta_1^2 + \beta_2^2 - 2\beta_1\beta_2$

$$= \frac{4\omega^2 v^2}{g^2} - \frac{4R\omega^2}{g}$$

$$= \frac{4\omega^2}{g^2}(v^2 - gR)$$

$$\Rightarrow \beta_2 - \beta_1 = \frac{2\omega}{g}\sqrt{v^2 - gR}$$

So distance $= 2R\sin\frac{\omega}{g}\sqrt{v^2-gR}$, as required.

STEP II 1999 Q12

$P(A) = 2p, \ P(B) = p, \ P(C) = 1-3p$ $\qquad 0 \le p \le 1/3$

$P(S|A) = 0.7, \ P(S|B) = 0.8, \ P(S|C) = 0.9$

$$P(A|S) = \frac{P(S|A)P(A)}{P(S)}$$

$$= \frac{0.7 \times 2p}{0.7 \times 2p + 0.8 \times p + 0.9 \times (1-3p)}$$

$$= \frac{14p}{14p + 8p + 9(1-3p)}$$

$$= \frac{14p}{9-5p}$$

$$P(C|S) = \frac{9-27p}{9-5p}$$

$$\frac{d(P(A|S))}{dp} = \frac{14(9-5p) + 5(14p)}{(9-5p)^2}$$

$$= \frac{126}{(9-5p)^2} > 0$$, so choose p maximal $\Rightarrow p = 1/3$.

This choice means A and B produce all the chips in the ratio 2:1, and C produces no chips.

$$\frac{d(P(C|S))}{dp} = \frac{-27(9-5p) + 5(9-27p)}{(9-5p)^2}$$

$$= \frac{-198}{(9-5p)^2} < 0$$, so choose p minimal $\Rightarrow p = 0$

This choice means C produces all chips and A and B produce none.

STEP II 1999 Q13

wlog, assume the cut happens on the first half of the stick, so the break point $K \sim U[0, \tfrac{1}{2}]$. Then $R = \frac{K}{1-K}$.

$$\text{So } P(R \le r) = P\left(\frac{K}{1-K} < r\right)$$
$$= P(K < r(1-K))$$
$$= P\left(K < \frac{r}{1+r}\right)$$
$$= \frac{2r}{1+r} \quad \text{for } r \in [0,1].$$

$$\text{So } f_R(r) = \frac{2(1+r)-2r}{(1+r)^2}$$
$$= \frac{2}{(1+r)^2}$$

$$ER = \int_0^1 \frac{2r}{(1+r)^2}\,dr = \int_0^1 \frac{-2}{(1+r)^2} + \frac{2}{(1+r)}\,dr$$

$$= \left[2(1+r)^{-1} + 2\ln|1+r|\right]_0^1$$

$$= \frac{2}{2} + 2\ln 2 - \frac{2}{1} - 2\ln 1$$
$$= 2\ln 2 - 1$$

$$ER^2 = \int_0^1 \frac{2r^2}{(1+r)^2}\,dr$$

$$= 2\int_0^1 1 - \frac{2r+1}{(1+r)^2}\,dr$$

$$= 2\int_0^1 1 - \frac{1}{(1+r)^2}\,dr - 2\int_0^1 \frac{2r}{(1+r)^2}\,dr$$

$$= 2\left[r + \frac{1}{1+r}\right]_0^1 - 2(2\ln 2 - 1)$$

$$= 2(1 + \tfrac{1}{2} - 1) - 2(2\ln 2 - 1)$$

$= 3 - 4\ln 2$

So $\text{Var} R = ER^2 - (ER)^2$

$= 3 - 4\ln 2 - (2\ln 2 - 1)^2$

$= 3 - 4\ln 2 - 4(\ln 2)^2 + 4\ln 2 - 1$

$= 2 - 4(\ln 2)^2$

STEP II 1999 Q14

Because the future of the game depends only on the current state, and not on how we got here, the only strategies are to stop playing after rolling a certain number or higher. Then the possible outcomes (1, 5, and 6 for stopping at ≥ 5, for example), are equally likely.

Stop at 6 : $\frac{1}{2}(1+6) = 7/2$

≥ 5 $\quad \frac{1}{3}(1+5+6) = 4$

≥ 4 $\quad \frac{1}{4}(1+4+5+6) = 4$

≥ 3 $\quad \frac{1}{5}(1+3+4+5+6) = 19/5$

≥ 2 $\quad \frac{1}{6}(1+2+3+4+5+6) = 7/2$

So the optimal strategy is to stop when rolling a 4 or higher, or when rolling a 5 or higher. Both have expected value 4.