STEP I 1999 Q1

There are 500,000 numbers which are divisible by 2, and 200,000 which are divisible by 5. However, we have counted 100,000 numbers twice (those which are divisible by 10)

So there are $1{,}000{,}000 - 500{,}000 - 200{,}000 + 100{,}000 = 400{,}000$ integers not divisible by 2 or 5.

Note 500,000 is divisible by both 2 and 5, so the multiples of both 2 and 5 occur in pairs with the mean of each pair being 500,000, as do the multiples of 10, and hence so too do the non-multiples. So the mean of all the numbers is 500,000

We have $4179 \div 3 = 1393$, $4179 \div 7 = 597$, $\frac{4179}{21} = 199$.

So, using the same argument as before, there are

$$4179 - 1393 - 597 + 199 = 1791$$

integers not divisible by 3 or 7.

Consider $\frac{4179}{2} = 2089.5$. This is halfway between the multiples of 3 2088 & 2091, the multiples of 7 2086 & 2093, and the multiples of 21 2079 and 2100. So, as before, the mean of the non-multiples is $4179/2$.

STEP I 1999 Q2

We have

$$(x-x_1)^2+(x-x_2)^2+(x-x_3)^2+(y-y_1)^2+(y-y_2)^2+(y-y_3)^2=a^2$$

Throughout, we only write the x terms for convenience - the y terms are exactly the same. We will add them in again at the end.

$$3x^2-2x(x_1+x_2+x_3)+(x_1^2+x_2^2+x_3^2)+ \_\_\_\_\_\_ = a^2$$

$$\Rightarrow x^2-\tfrac{2}{3}x(x_1+x_2+x_3)^2+\_\_\_\_ = \tfrac{1}{3}(a^2-(x_1^2+x_2^2+x_3^2)+\_\_\_)$$

$$\Rightarrow (x-\tfrac{1}{3}(x_1+x_2+x_3))^2-\tfrac{1}{9}(x_1+x_2+x_3)^2+\_\_\_ = \tfrac{1}{3}(a^2-(x_1^2+x_2^2+x_3^2)+\_\_\_)$$

$$\Rightarrow (x-\tfrac{1}{3}(x_1+x_2+x_3))^2+\_\_\_\_ = \tfrac{1}{3}(a^2-(x_1^2+x_2^2+x_3^2)+\tfrac{1}{3}(x_1+x_2+x_3)^2+\_\_\_) \qquad (*)$$

Now, note that

$$\begin{aligned} d^2 &= (x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\_\_\_ \\ &= x_1^2+x_2^2+x_3^2+3\bar{x}^2-2\bar{x}(x_1+x_2+x_3)+\_\_\_ \\ &= x_1^2+x_2^2+x_3^2+\tfrac{1}{3}(x_1+x_2+x_3)^2-\tfrac{2}{3}(x_1+x_2+x_3)^2+\_\_\_ \\ &= x_1^2+x_2^2+x_3^2-\tfrac{1}{3}(x_1+x_2+x_3)^2+\_\_\_ \end{aligned}$$

where d^2 is the sum of the squared distances from the three points to the centroid.

Then $(*)$ becomes

$$(x-\bar{x})^2+(y-\bar{y})^2=\tfrac{1}{3}(a^2-d^2)$$

Which is a circle of centre $\bar{x}$ and radius $\sqrt{\tfrac{1}{3}(a^2-d^2)}$.

For the radius to be exist, we must have $a^2>d^2$.

STEP I 1999 Q3

(i) Suppose $x_i \leq 1$ for some i. Then $1+\frac{1}{x_{i+1}} \leq 1 \Rightarrow \frac{1}{x_{i+1}} \leq 0$

$\Rightarrow x_{i+1} \leq 0$, a contradiction, as $x_k \geq 0$ for all k.

(ii) $x_1 - x_2 = 1 + \frac{1}{x_2} - \left(1+\frac{1}{x_3}\right)$

$= \frac{1}{x_2} - \frac{1}{x_3}$

$= \frac{x_3 - x_2}{x_2 x_3}$

$= \frac{-(x_2 - x_3)}{x_2 x_3}$

(iii) $x_1 - x_2 = \frac{-(x_2-x_3)}{x_2 x_3}$

$= \frac{x_3 - x_4}{x_2 x_3^2 x_4}$

$= \frac{-(x_4 - x_5)}{x_2 x_3^2 x_4^2 x_5}$

$= \dots$

$= \frac{(-1)^n (x_1 - x_2)}{x_2^2 x_3^2 \text{---} x_1^2}$

$\Rightarrow \frac{(-1)^n}{x_1^2 \text{---} x_n^2}$ or $x_1 = x_2$

$\Rightarrow x_1^2 \text{---} x_n^2 = (-1)^n$, which is not possible, as $x_i > 1$ for all i, so $x_i^2 \text{---} x_n^2 > 1$.

Hence $x_1 = x_2$, and by extension $x_1 = x_2 = x_3 \text{---} = x_n$.

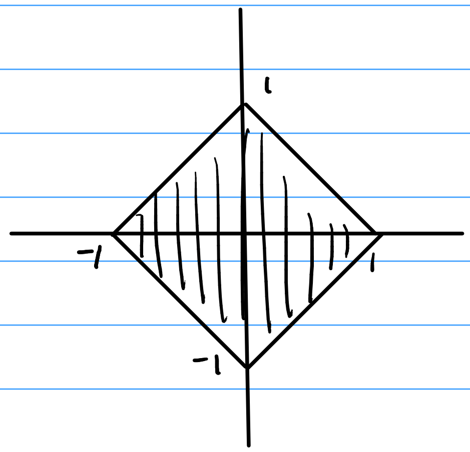
So $x_1 = 1 + \frac{1}{x_1}$

$\Rightarrow x_1^2 - x_1 - 1 = 0$

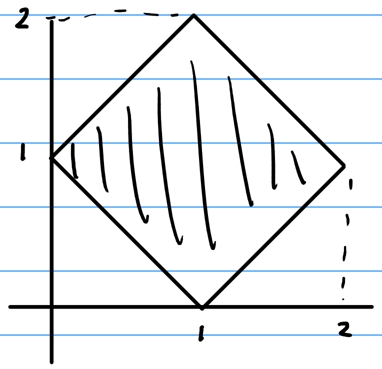
$\Rightarrow x_1 = \frac{1 \pm \sqrt{5}}{2}$ but $x_1 > 1$, so $x_1 = \frac{1+\sqrt{5}}{2}$

STEP I 1999 Q4

(i) $|x|+|y| \leq 1$

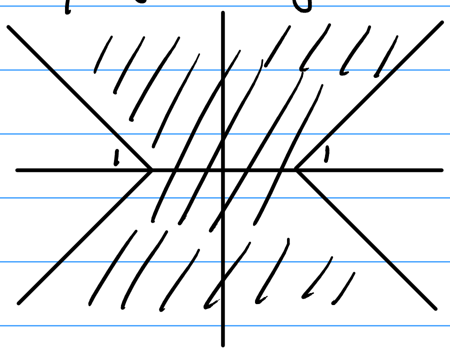


(ii) $|x-1|+|y-1| \leq 1$ is a translation of (i) by $\begin{pmatrix}1\\1\end{pmatrix}$

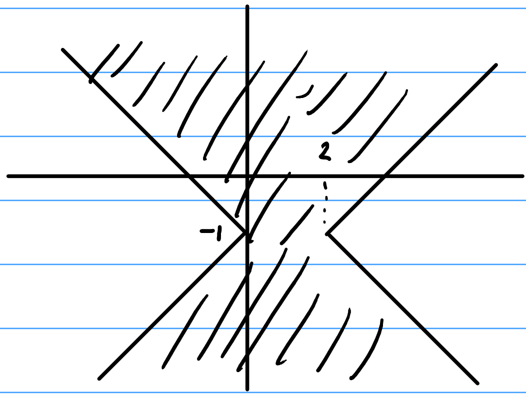


(iii) Start with $|x|-|y| \leq 1$

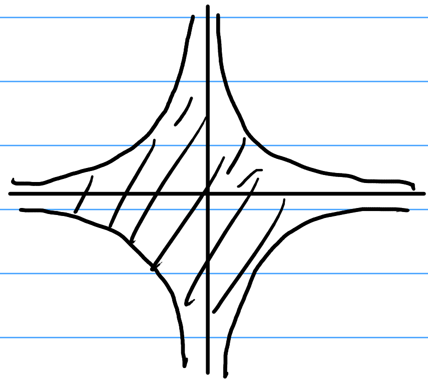
For $x>0, y>0$ the boundary is $x-y=1 \Rightarrow y=x-1$. The other quadrants are reflections of this. The origin satisfies the inequality so this region is shaded.



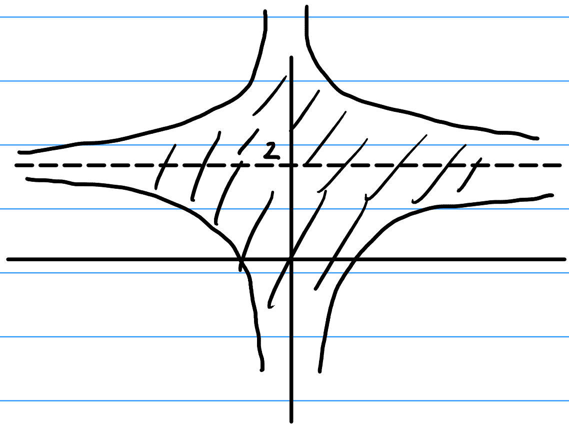
Then $|x-1|-|y+1| \leq 1$ is a translation of this by $\begin{pmatrix}1\\-1\end{pmatrix}$



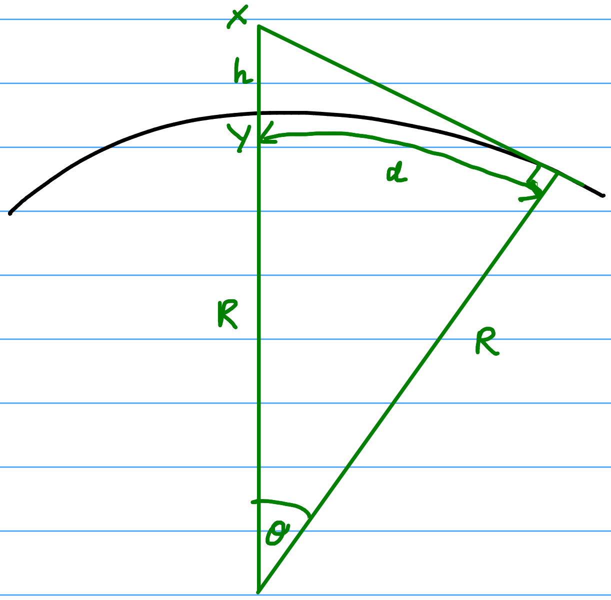
1(v) $|x||y| \leq 1$ is



$|x||y-2| \leq 1$ is a translation of this graph by the vector $\binom{0}{2}$.



STEP I 1999 Q5



(i) We have $\theta = \cos^{-1}\left(\frac{R}{R+h}\right)$, so $d = R\theta = R\cos^{-1}\left(\frac{R}{R+h}\right)$

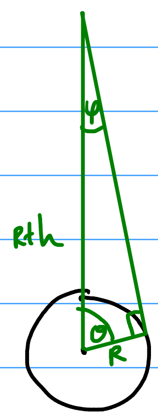
(ii) We have $\sin\theta = \frac{\sqrt{(R+h)^2 - R^2}}{R+h} = \frac{\sqrt{h^2+2Rh}}{R+h}$

But $h \ll R$, so $h^2 \ll 2Rh$, so $\sin\theta \approx \frac{\sqrt{2Rh}}{R}$

Further $h \ll R \Rightarrow 0 < \theta \ll 1$, so $\sin\theta \approx \theta$. Hence $d = R\theta \approx R \cdot \frac{\sqrt{2Rh}}{R}$

$= \sqrt{2RH}$, so $k = \sqrt{2}$.

(iii)



we have $\varphi \approx \sin\varphi = \frac{R}{R+h} \approx \frac{R}{h}$

$\theta = \pi/2 - \varphi \approx \pi/2 - R/h$

So $d = R\theta \approx \frac{\pi}{2}R - \frac{R^2}{h}$ so $a = \pi/2$, $b = -1$

STEP I 1999 Q6

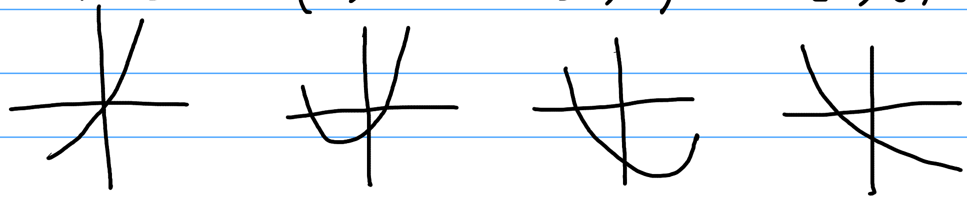
(i) $b > 0$ $\quad$ $\min = -10b + a$ $\quad$ $\max = 10b + a$

$b = 0$ $\quad$ $\min = \max = a$

$b < 0$ $\quad$ $\min = 10b + a$ $\quad$ $\max = -10b + a$

(ii) If $c = 0$, then we have a repeat of (i). Otherwise, consider the x coordinate of the minimum point which is $x = -b/2c$. Here, $cx^2 + bx + a = \frac{-b^2}{4c} + a$

$\frac{-b}{2c} \in$ $\quad$ $(-\infty, -10]$ $\quad$ $(-10, 0)$ $\quad$ $[0, 10)$ $\quad$ $[10, \infty)$



So,

$-b/2c$	minimum	maximum
$(-\infty, -10]$	$100c - 10b + a$	$100c + 10b + a$
$(-10, 0)$	$a - b^2/4c$	$100c + 10b + a$
$[0, 10)$	$a - b^2/4c$	$100c - 10b + a$
$[10, \infty)$	$100c + 10b + a$	$100c - 10b + a$

STEP I 1999 Q7

$y = \sin(k\arcsin x)$

$\frac{dy}{dx} = \cos(k\arcsin x) \cdot k(1-x^2)^{-1/2}$

$\frac{d^2y}{dx^2} = -\sin(k\arcsin x) \cdot k^2(1-x^2)^{-1} + x\cos(k\arcsin x) \cdot k(1-x^2)^{-3/2}$

Substituting into (*),

$-k^2\sin(k\arcsin x) + kx\cos(k\arcsin x)(1-x^2)^{-1/2} - kx\cos(k\arcsin x) \cdot k(1-x^2)^{-1/2} + k^2\sin(k\arcsin x)$

$= 0$, as required.

Now $k=3$, $y = Ax^3 + Bx^2 + Cx + D$

$y(0) = 0 \Rightarrow D = 0$

$y' = 3Ax^2 + 2Bx + C$

$y'(0) = 3 \Rightarrow C = 3$

$y'' = 6Ax + 2B$

Substituting into (*),

$(1-x^2)(6Ax+2B) - x(3Ax^2+2Bx+3) + 9(Ax^3+Bx^2+3x) = 0$

$\Rightarrow x^3(-6A-3A+9A) + x^2(2B-2B+9B) + x(6A-3+27) + (2B) = 0$

$\Rightarrow 9Bx^2 + (24+6A)x + 2B = 0$

$\Rightarrow B=0,\ A=-4$

So $y = -4x^3 + 3x$

Let $x = \sin\theta$, then $y = 3\sin\theta - 4\sin^3\theta$

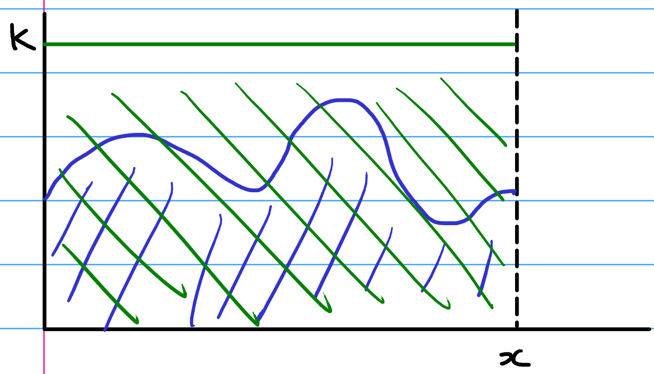
But also $y = \sin(3\sin^{-1}(\sin\theta))$

$= \sin 3\theta$ is a solution of $(*)$

Further, $\left.\begin{array}{l} \sin(3\theta) = 0 \\ \frac{d}{d\theta}\sin(3\theta) = 3\cos 3\theta = 3 \end{array}\right\}$ at $\theta = 0$

So $y = \sin 3\theta$ and $y = -4\sin^3\theta + \sin\theta$ both satisfy $(*)$ subject to the same initial conditions, so are the same function.

STEP I 1999 Q8



By comparison of areas, we see that

$$0 \le \int_0^x f(t)\,dt \le kx.$$

Now consider $f(t) = \frac{t}{n(n-t)}$ and $k=1$, with $n>1$. Clearly $0 \le f(t)$, and further $\frac{t}{n(n-t)}$ is increasing in t so its maximum on $[0,1]$ is $f(1) = \frac{1}{n(n-1)} = \frac{1}{n} - \frac{1}{n-1}$

So, $$0 \le \int_0^1 \frac{t}{n(n-t)}\,dt \le \frac{1}{n} - \frac{1}{n-1}$$

Now $$\int_0^1 \frac{t}{n(n-t)}\,dt = \frac{1}{n}\int_0^1 \frac{t}{n-t}\,dt$$

$$= \frac{1}{n}\int_0^1 \frac{t-n+n}{n-t}\,dt$$

$$= \frac{1}{n}\int_0^1 -1 + \frac{n}{n-t}\,dt$$

$$= \frac{1}{n}\Big[-t - n\ln|n-t|\Big]_0^1$$

$$= \frac{1}{n}\left(-1 - n\ln(n-1) + n\ln n\right)$$

$$= \frac{-1}{n} + \ln\left(\frac{n}{n-1}\right)$$

So, $0 \le \ln\left(\frac{n}{n-1}\right) - \frac{1}{n} \le \frac{1}{n-1} - \frac{1}{n}$, as required.

Summing from 2 to N,

$$0 \le \sum_{n=2}^{N}\left[\ln(n) - \ln(n-1)\right] - \sum_{n=2}^{N}\frac{1}{n} \le \sum_{n=2}^{N}\frac{1}{(n-1)} - \frac{1}{n}$$

$$\Rightarrow \quad 0 \le \ln N - \ln 1 - \sum_{n=2}^{N}\frac{1}{n} \le 1 - \frac{1}{N}$$

$$\Rightarrow \quad 0 \le \ln N - \sum_{n=2}^{N}\frac{1}{n} \le 1$$

Taking the limit as $N \to \infty$, $\ln N \to \infty$ so we must have $\sum_{n=2}^{N}\frac{1}{n} \longrightarrow \infty$ as $\ln N - \sum_{n=2}^{N}\frac{1}{n} \le 1 \Rightarrow \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n} = \lim_{N\to\infty}\left(1 + \sum_{n=2}^{N}\frac{1}{n}\right) = \infty$.

We have $\sum_{n=2}^{N}\frac{1}{n} \le \ln N$

$$\begin{aligned}
\Rightarrow \quad \sum_{n=1}^{10^{30}}\frac{1}{n} &\le 1 + \ln(10^{30}) \\
&= \ln(1000^{10}) + 1 \\
&< \ln(1024^{10}) + 1 \\
&= \ln(2^{100}) + 1 \\
&< 100 + 1 \qquad \text{as } e > 2 \\
&= 101
\end{aligned}$$

So $\sum_{n=1}^{10^{30}}\frac{1}{n} < 101$, and so clearly $\sum_{n=1}^{N}\frac{1}{n} < 101$ for $N < 10^{30}$, as $\frac{1}{n} > 0$ for $n > 0$.

STEP I 1999 Q9

After 0.5km, the tortoise has been travelling for $\frac{0.5}{v}$ and the hare for $0.5+\frac{0.5}{V}$

$\Rightarrow \frac{0.5}{v}=0.5+\frac{0.5}{V} \Rightarrow \frac{1}{v}=1+\frac{1}{V}$ (✱)

At the second meeting, the tortoise has travelled $x-1.25$ and the hare $x+1.25$, so

$$\frac{x-1.25}{v}=\frac{x+1.25}{V}+1 \qquad (\dagger)$$

Substituting (✱) into (†),

$$x-1.25\left(1+\frac{1}{V}\right)=\frac{x+1.25}{V}+1$$

$\Rightarrow (x-1.25)(V+1)=x+1.25+V$

$\Rightarrow (4x-5)(V+1)=4x+5+4V$

$\Rightarrow 4xV+4x-5V-5=4x+5+4V$

$\Rightarrow V(4x-9)=10$

$\Rightarrow V=\frac{10}{4x-9}$

$\frac{1}{v}=1+\frac{1}{V}=1+\frac{4x-9}{10}$

$=\frac{4x+1}{10}$

$\Rightarrow v=\frac{10}{4x+1}$

We have $\frac{2x}{V}=0.5$ (0.5 hours travel time and 1 hour sleeping)

$\Rightarrow 2x=0.5\left(\frac{10}{4x-9}\right)$

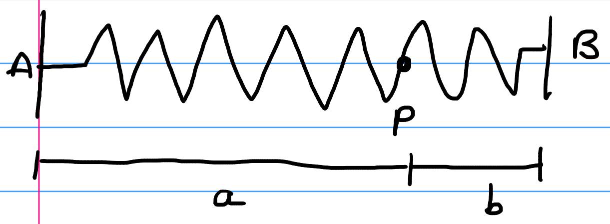
$\Rightarrow 2x(4x-9)=5$

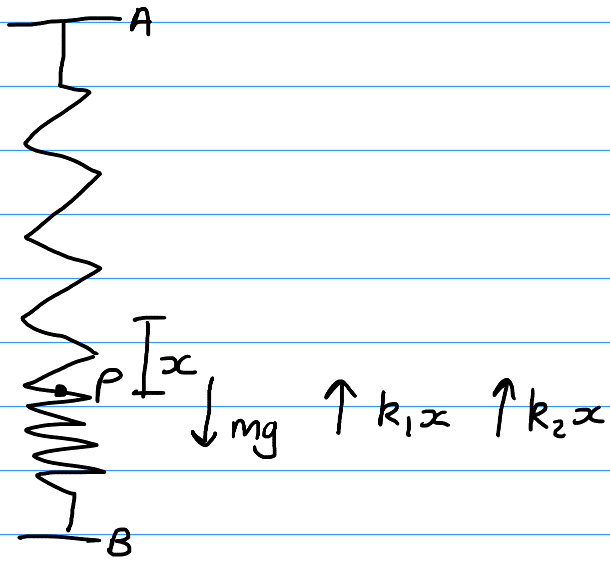
$\Rightarrow 8x^2-18x-5=0$

$\Rightarrow (2x-5)(4x+1)$

$\Rightarrow x=\frac{5}{2}$ km (as $x>0$)

STEP I 1999 Q10





x is the displacement from equilibrium, and k_1 and k_2 are the spring constants of spring AP and BP respectively.

So, $m\ddot{x} = mg - x(k_1 + k_2)$
$= mg - x\left(\frac{\lambda}{a} + \frac{\lambda}{b}\right)$
$= mg - \lambda x\left(\frac{a+b}{ab}\right)$

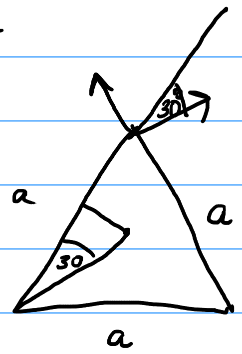
$\Rightarrow \ddot{x} = -x\frac{\lambda}{m}\left(\frac{a+b}{ab}\right) + g$

$\Rightarrow x = A\sin(\omega x) + B\sin(\omega x) + \frac{gm}{\lambda}\left(\frac{ab}{a+b}\right)$

where $\omega = \sqrt{\frac{\lambda}{m}\left(\frac{a+b}{b}\right)}$. This is simple harmonic motion, independant of 2.

STEP I 1999 Q11

$F = \frac{\gamma m_1 m_2}{r^2}$



The net force on each star is $\frac{2\gamma m^2 \cos(30^\circ)}{a^2}$

$$= \frac{\gamma m^2}{a^2}\sqrt{3}$$

The distance to the CoM is $\frac{a}{2} \div \cos(30^\circ) = \frac{a}{\sqrt{3}}$

$F = m\omega^2 r \Rightarrow \omega^2 = \frac{F}{mr} = \frac{\gamma m^2}{a^2}\sqrt{3} \times \frac{1}{m} \times \frac{\sqrt{3}}{a}$

$$= \frac{3\gamma m}{a^2}$$

$$\Rightarrow \omega = \sqrt{\frac{3\gamma m}{a^2}}$$

Now $F = \frac{\gamma m^2}{a^2}\sqrt{3} + \frac{\gamma \lambda m^2}{(a/\sqrt{3})^2}$

$$= \frac{\gamma m^2}{a^2}\left(\sqrt{3} + 3\lambda\right)$$

So $\omega^2 = \frac{\gamma m^2}{a^2}\left(\sqrt{3}+3\lambda\right) \times \frac{1}{m} \times \frac{\sqrt{3}}{a}$

$$= \frac{\gamma m}{a^3}\left(3 + 3\sqrt{3}\lambda\right)$$

$$\Rightarrow \omega = \sqrt{\frac{\gamma m}{a^3}\left(3+3\sqrt{3}\lambda\right)}$$

STEP I 1999 Q12

(i) $x + \frac{1}{x} \geqslant 2$

$\Leftrightarrow x^2 - 2x + 1 \geqslant 0$ (as $x > 0$)

$\Leftrightarrow (x-1)^2 \geqslant 0$, which is true.

$P(2) = q_1 r_1$

$P(12) = q_6 r_6$

$P(7) = q_1 r_6 + q_2 r_5 + \text{———} + q_6 r_1$

$\geqslant q_1 r_6 + q_6 r_1$

$= \frac{q_1 r_1 r_6}{r_1} + \frac{q_6 r_6 r_1}{r_6}$

$= p(2)\frac{r_6}{r_1} + p(12)\frac{r_1}{r_6}$

wlog $p(2) \geqslant p(12)$.

Then $p(2)\frac{r_6}{r_1} + p(12)\frac{r_1}{r_6}$

$\geqslant p(12)\left(\frac{r_6}{r_1} + \frac{r_1}{r_6}\right)$

$= p(12)\left(x + \frac{1}{x}\right)$ (where $x = \frac{r_6}{r_1} > 0$)

$\geqslant 2p(12)$ by the first inequality

$> p(12)$.

So $p(7) > p(12)$, so not all outcomes are equally likely.

(ii) At least one $q_j \geqslant \frac{1}{6}$ (as $q_j \geqslant 0$, $\sum q_j = 1$).

Then $P(2 \text{ numbers the same}) = \sum q_j^2 \geqslant (1/6)^2$

$= 1/36$.

So this probability cannot be less than $1/36$.

STEP I 1999 Q13

After the first magnet, the others join to the current unit with probability $1/2$, and form a new one with probability $1/2$. Thus, the number of units is distributed $1 + B(N-1, 1/2)$. This has expectation $1 + \frac{1}{2}(N-1) = \frac{1}{2}(N+1)$, and variance $\frac{1}{4}(N-1)$.

STEP I 1999 Q14

(i) We have $E(\text{score}) = 4\int_0^{1/4} 2x\,dx + 3\int_{1/4}^{1/2} 2x\,dx + 2\int_{1/2}^{3/4} 2x\,dx + \int_{3/4}^{1} 2x\,dx$

$$= 4(1/4)^2 + 3\left(\frac{4}{16} - \frac{1}{16}\right) + 2\left(\frac{9}{16} - \frac{4}{16}\right) + \left(\frac{16}{16} - \frac{9}{16}\right)$$

$$= \frac{4}{16} + \frac{9}{16} + \frac{10}{16} + \frac{7}{16}$$
$$= \frac{30}{16}$$
$$= \frac{15}{8}$$

(ii) Ways of winning:

3	$3/16$	$= 768/4096$
2 1	$5/16 \times 7/16$	$= 560/4096$
1 2	$7/16 \times 5/16$	$= 560/4096$
1 1 1	$(7/16)^3$	$= 343/4096$

So $P(1,1,1 \mid \text{won}) = \dfrac{343}{768+560+560+343}$

$$= \frac{343}{2231}$$