STEP III 1998 Q1

$f(x) = \sin^2 x + 2\cos x + 1$

$f'(x) = 2\sin x\cos x - 2\sin x = 0$

$\Rightarrow \sin x(\cos x - 1) = 0$

$\Rightarrow x = 0, \pi, 2\pi$

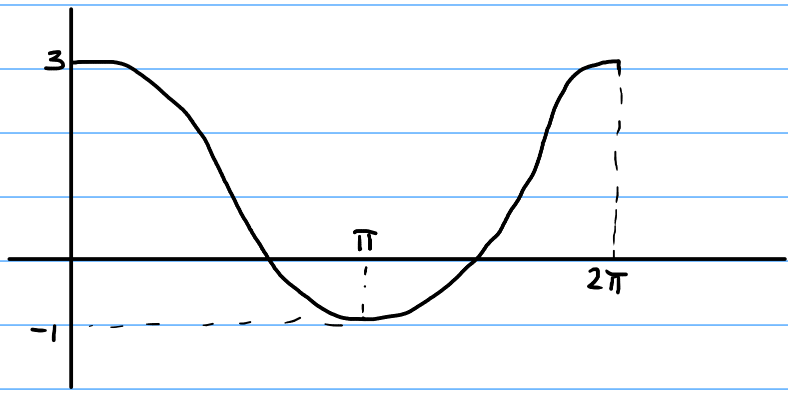
$f''(x) = 2\cos^2 x - 2\sin^2 x - 2\cos x$

$f''(0) = 0$

$f''(\pi) = 4$ so minimum point. As the function is 2π-periodic, $x = 0, 2\pi$ must be maximum points.

$f(0) = f(2\pi) = 3$

$f(\pi) = -1$



$$g(x) = \frac{af(x)+b}{cf(x)+d}$$

$$g'(x) = \frac{af'(x)(cf(x)+d) - cf'(x)(af(x)+b)}{(cf(x)+d)^2} = 0$$

$\Leftrightarrow f'(x)\left(acf(x) + ad - acf(x) - bc\right) = 0$

$\Leftrightarrow f'(x)(ad - bc) = 0$

but $ad \neq bc$ so $f'(x) = 0$

So the stationary points of g occur at the same x values as those of f.

$$g(0) = g(2\pi) = \frac{3a+b}{3c+d}$$

$$g(\pi) = \frac{-a+b}{-c+d}$$

For $|g(x)|$ to be arbitrarly large, we need $cf(x)+d$ to get arbitrarily close to zero. Suppose $cf(x)+d=0$, then $f(x) = -d/c$.

But $-1 \leq f(x) \leq 3$, so to make $|g(x)|$ arbitrarily large, we must have $d/c \in [-3,1]$.

STEP III 1998 Q2

$I(a,b) = \int_0^1 t^a(1-t)^b\,dt$

(i) $I(a,b) = \int_0^1 t^a(1-t)^b\,dt$ set $u = 1-t$, $du = -dt$

$= \int_1^0 (1-u)^a u^b(-du)$

$= \int_0^1 u^b(1-u)^a\,du$

$= I(b,a)$

(ii) $I(a+1,b) + I(a,b+1) = \int_0^1 t^{a+1}(1-t)^b + t^a(1-t)^{b+1}\,dt$

$= \int_0^1 t^a(1-t)^b(t+1-t)\,dt$

$= \int_0^1 t^a(1-t)^b\,dt$

$= I(a,b)$

(iii) $(a+1)I(a,b) = (a+1)\int_0^1 t^a(1-t)^b\,dt$ $\quad u \;\; (1-t)^b \quad v' \;\; t^a$

$u' \;\; -b(1-t)^{b-1} \quad v \;\; \frac{1}{a+1}t^{a+1}$

$= (a+1)\underbrace{\left[\frac{t^{a+1}(1-t)^b}{a+1}\right]_0^1}_{=0} + (a+1)\int_0^1 \frac{b}{a+1}(1-t)^{b-1}t^{a+1}\,dt$

$= b\int_0^1 t^{a+1}(1-t)^{b-1}\,dt$

$= bI(a+1, b-1)$, as required.

we have $I(a,b) = \frac{b}{a+1} I(a+1, b-1)$

$$= \frac{b(b-1)}{(a+1)(a+2)} I(a+2, b-2)$$

$$= \dots$$

$$= \frac{b(b-1)(b-2)\cdots(1)}{(a+1)(a+2)\cdots(a+b)} I(a+b, 0)$$

$$= \frac{b!}{(a+b)!/a!} I(a+b, 0)$$

$$= \frac{a!b!}{(a+b)!} \int_0^1 t^{a+b} \, dt$$

$$= \frac{a!b!}{(a+b)!} \cdot \left[\frac{t^{a+b+1}}{a+b+1} \right]_0^1$$

$$= \frac{a!b!}{(a+b+1)!}$$

STEP III 1998 Q3

$V_{N+1} = (1+c)V_N - d \qquad (*)$

Set $V_T = Ak^T + B$. Substituting into $(*)$,

$$Ak^{N+1} + B = (1+c)(Ak^N + B) - d$$
$$\Rightarrow Ak \cdot k^N + \cancel{B} = (1+c)Ak^N + \cancel{B} + cB - d$$

And so $k = 1+c$ and $cB - d = 0$
$$\Rightarrow B = d/c$$

So $V_N = A(1+c)^N + d/c$.

But $V_0 = A + d/c \Rightarrow A = V_0 - d/c$, so

$$V_N = (V_0 - d/c)(1+c)^N + d/c \qquad (\dagger)$$

If $c=0$, $(*)$ becomes $V_{N+1} = V_N - d$
$$\Rightarrow V_N = V_0 - Nd$$

Using $(\dagger)$, $\lim_{c\to 0} V_N = \lim_{c \to 0} (V_0 - d/c)(1+c)^N + d/c$

$$= \lim_{c\to 0} (V_0 - d/c)(1+Nc) + d/c$$

$$= \lim_{c\to 0} V_0 + NcV_0 - d/c - Nd + d/c$$

$$= \lim_{c\to 0} V_0 - Nd + cNV_0$$

$$= V_0 - Nd, \text{ as required.}$$

STEP III 1998 Q4

$r = \cos\theta$

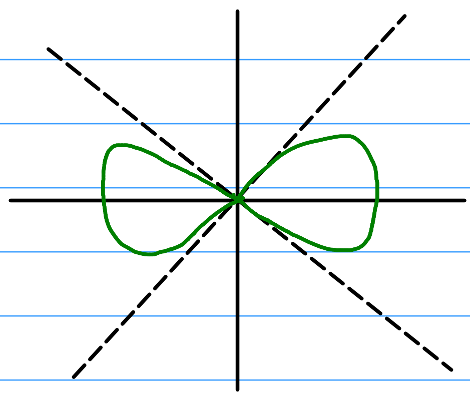
$\Rightarrow r^2 = r\cos\theta$

$\Rightarrow x^2 + y^2 = x$

$\Rightarrow (x - 1/2)^2 + y^2 = 1/4$

Further, this circle exists entirely to the right of the y-axis (and at the origin), so we have $-\pi/2 < \theta < \pi/2 \Rightarrow r = \cos\theta > 0$.

$r = \cos(2\theta)$ (plotting only for $r > 0$)



$r = \cos(2n\theta)$ has $2n$ loops evenly spaced around the circle, each of width $\frac{\pi}{2n}$, and one centred at $\theta = 0$.

We find the area of half of one of these loops and multiply by $4n$.

$$A = 4n \int \tfrac{1}{2} r^2 \, d\theta$$

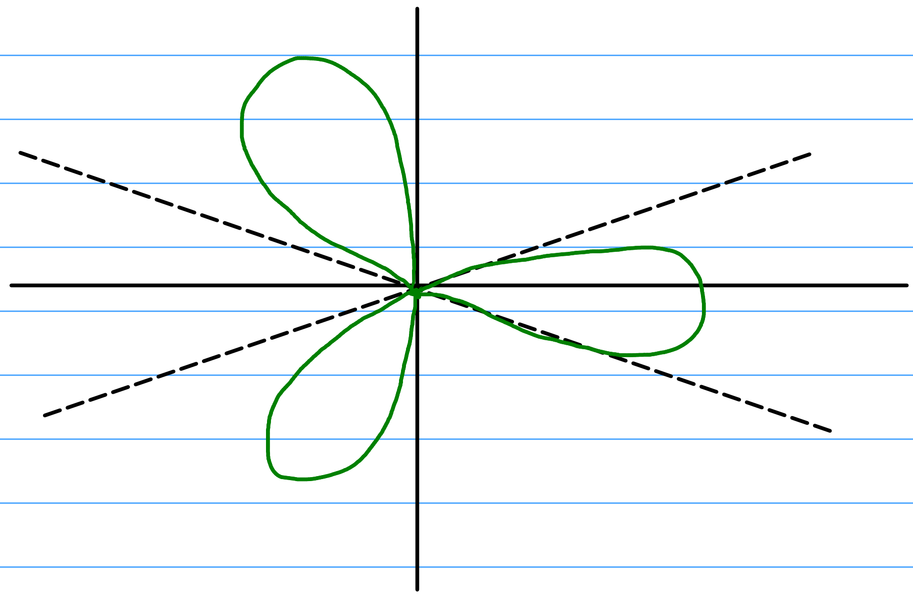
$$= 4n \int_0^{\frac{\pi}{2n}} \tfrac{1}{2} \cdot \tfrac{1}{2} (1 - \cos(4n\theta)) \, d\theta$$

$$= n \left[\theta - \frac{1}{4n} \sin(4n\theta) \right]_0^{\pi/2n}$$

$$= n \left(\frac{\pi}{2n} \right)$$

$= \frac{\pi}{2}$, which is independent of r.

$\cos 3\theta$ is positive for $-\pi/6 < \theta < \pi/6$, $\pi/2 < \theta < 5\pi/6$, $7\pi/6 < \theta < 3\pi/2$



STEP III 1998 Q5

$M^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$

$M^3 = -M$

$M^4 = I$, and then it repeats.

$\exp(\theta M) = I + \theta M - \frac{\theta^2}{2!}I - \frac{\theta^3}{3!}M + \frac{\theta^4}{4!} + \frac{\theta^5}{5!}M + \text{—}$

$= I(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \text{—}) + M(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \text{—})$

$= I\cos\theta + M\sin\theta$

$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ which is a rotation by θ anticlockwise about the origin.

$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $N^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $N^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $k \geqslant 2$

So $\exp(sN) = I + sN$

$= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ which is a shear parallel to the x-axis by factor s.

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & s\cos\theta - \sin\theta \\ \sin\theta & s\sin\theta + \cos\theta \end{pmatrix}$

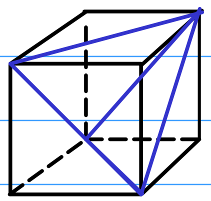
$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta + s\sin\theta & s\cos\theta - \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

For these to be equal, we must have $s\sin\theta = 0$ for all s $\Rightarrow \theta = 0, \pi\ (, 2\pi, 3\pi, \ldots)$.

So the rotation must keep the x axis parallel to where it started

STEP III 1998 Q6

(i) Pick one vertex of the cube. Suppose we then picked the vertex furtherst from this first vertex. Then all remaining vertices are adjacent to these two, so we cannot pick the two more vertices we need. Instead, we must have (a possible transformation of)

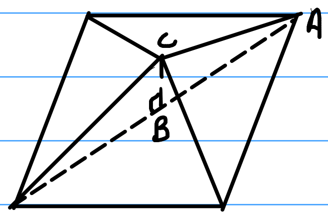


If the tetrahedron, then the cube has side length $1/\sqrt{2}$, so the volume of the cube is $\left(1/\sqrt{2}\right)^3 = \sqrt{2}/4$.

There are 4 pyramids in the cube not in the tetrahedron, each with volume $\frac{1}{3}\times\left(\frac{1}{2}\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\right)\times\frac{1}{\sqrt{2}} = \frac{1}{6}\cdot\frac{\sqrt{2}}{4}$.

So the volume of the tetrahedron is $\frac{\sqrt{2}}{4} - 4\cdot\frac{1}{6}\frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{12}$

(ii)



AB has length $\sqrt{2}/2$, AC has length 1. So BC has length $\sqrt{1^2-\left(\sqrt{2}/2\right)^2} = \sqrt{2}/2$. So the volume of the pyramid is $\frac{1}{3}\times1\times1\times\frac{\sqrt{2}}{2} = \sqrt{2}/6$. So the volume of the octahedron is $2\times\sqrt{2}/6 = \sqrt{2}/3$

(iii) Suppose the cube has side length a. Then the tetrahedron has side length $\sqrt{2}a$, so has volume $(\sqrt{2}a)^3\cdot\frac{\sqrt{2}}{12} = a^3/3$

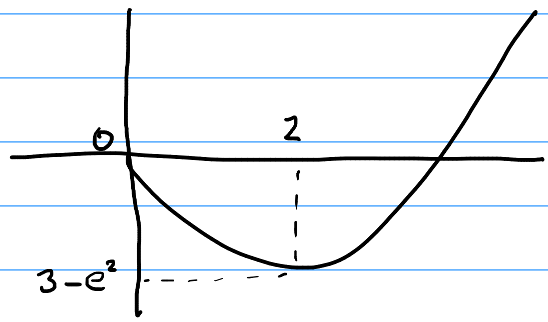
The side length of the octahedron is $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}a$, so the octahedron has volume $\left(\frac{\sqrt{2}}{2}a\right)^3\cdot\frac{\sqrt{2}}{3} = a^3/6$, which is half the volume of the tetrahedron.

STEP III 1998 Q7

$f(s) = e^s(s-3)+3$

$f'(s) = e^s(s-3+1) = 0$

$\Rightarrow s=2$



If $e \approx 2.7$, $f(1) = 2.7 \times -2 + 3 < 0$

$f(2) = -2.7^2 + 3 < 0$

$f(3) = 3 > 0$

So $m=3$.

$$b(x) = \frac{x^3}{e^{x/T}-1}$$

$$b'(x) = \frac{3x^2(e^{x/T}-1) - \frac{1}{T}e^{x/T}x^3}{(e^{x/T}-1)^2} = 0$$

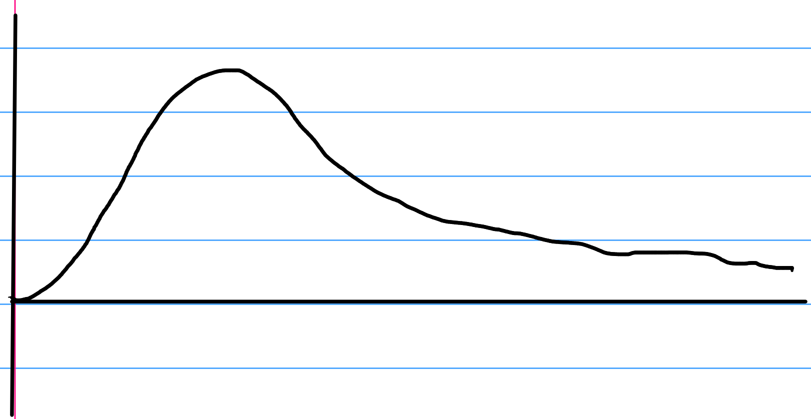
$$\Rightarrow x^2\left[e^{x/T}(3-\tfrac{x}{T}) - 3\right] = 0$$

$$\Rightarrow e^{x/T}(\tfrac{x}{T}-3)+3=0$$

Set $s = x/T$ (and $x \in (0,\infty) \Rightarrow s = x/T \in (0,\infty)$), then there is one root between 2 and 3, so one turning point.

For small x, $b(x) \approx \frac{x^3}{1+x/T-1}$
$= Tx^2$

For large x, $b(x) \to 0$ as $e^{x/T} \gg x^3$



Suppose $\int_0^\infty b(x)\,dx = B < \infty$.

Now $\int_0^\infty \frac{x^3}{e^{x/T}-1}\,dx$ set $u = x/T$, $du = \frac{1}{T}dx$

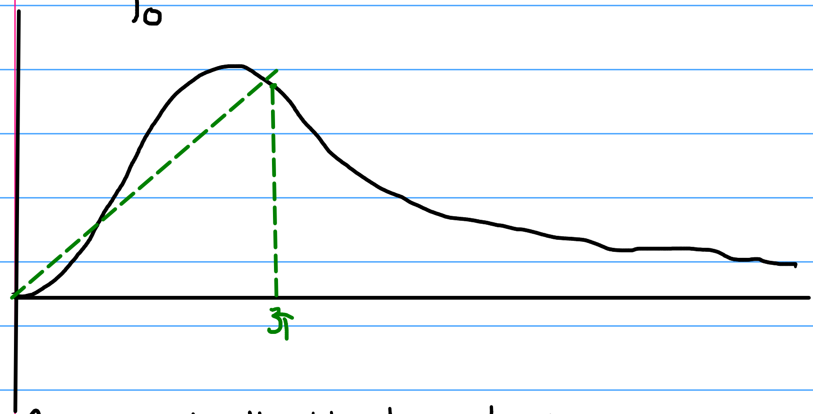
$$= \int_0^\infty \frac{T^3u^3}{e^u-1}T\,du$$

$$= \left(\int_0^\infty \frac{u^3}{u-1}\,du\right)T^4$$

$$= kT^4$$

$$B \approx 2\int_0^{T_M} b(x)\,dx$$

$$= 2\int_0^{3T} b(x)\,dx$$



Approximating the integral as a triangle,

$$B \approx 2\cdot\tfrac{1}{2}\cdot 3T\cdot b(3T)$$

$$= 3T\cdot\frac{(3T)^3}{e^3-1}$$

$$= \frac{81}{e^3-1}T^4$$

So $k \approx \frac{81}{e^3-1}$

STEP III 1998 Q8

(i) $\mathbf{r} = \mathbf{b} + \lambda\mathbf{m}$ and $\mathbf{r}\cdot\mathbf{r} = a^2$

So $(\mathbf{b}+\lambda\mathbf{m})\cdot(\mathbf{b}+\lambda\mathbf{m}) = a^2$

$\Rightarrow \quad \mathbf{b}\cdot\mathbf{b} + 2\lambda\mathbf{b}\cdot\mathbf{m} + \lambda^2\mathbf{m}\cdot\mathbf{m} = a^2$

$\Rightarrow \lambda^2 + 2\lambda\mathbf{b}\cdot\mathbf{m} + \mathbf{b}\cdot\mathbf{b} - a^2 = 0 \qquad$ as $\mathbf{m}\cdot\mathbf{m} = 1$

this has two solutions, so

$$(2\mathbf{b}\cdot\mathbf{m})^2 - 4(\mathbf{b}\cdot\mathbf{b} - a^2) > 0$$

$\Rightarrow \quad (\mathbf{b}\cdot\mathbf{m})^2 - \mathbf{b}\cdot\mathbf{b} + a^2 > 0$

$\Rightarrow \quad a^2 > \mathbf{b}\cdot\mathbf{b} - (\mathbf{b}\cdot\mathbf{m})^2$

There is exactly one point of intersection if $a^2 = \mathbf{b}\cdot\mathbf{b} - (\mathbf{b}\cdot\mathbf{m})^2$. In this case,

$\lambda^2 + 2\lambda(\mathbf{b}\cdot\mathbf{m}) + (\mathbf{b}\cdot\mathbf{m})^2 = 0$

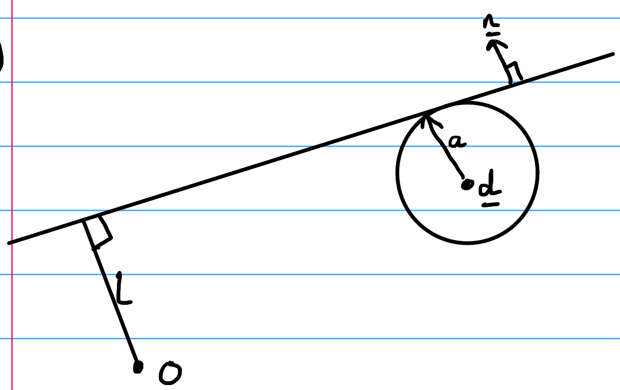
$\Rightarrow \lambda = -\mathbf{b}\cdot\mathbf{m}$

So $\mathbf{p} = \mathbf{b} + \mathbf{m}(-\mathbf{b}\cdot\mathbf{m})$, hence $\mathbf{p}\cdot\mathbf{m} = \mathbf{b}\cdot\mathbf{m}\cdot(\mathbf{m}\cdot\mathbf{m})(-\mathbf{b}\cdot\mathbf{m})$

$= \mathbf{b}\cdot\mathbf{m} - \mathbf{b}\cdot\mathbf{m}$

$= 0$

(ii)



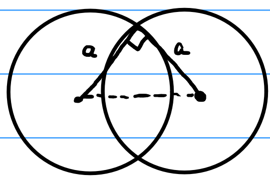
We can see from the diagram that for the plane to be tangential to the sphere we need $\mathbf{d} + a\mathbf{n}$ or $\mathbf{d} - a\mathbf{n}$ on the plane. The plane has equation $\mathbf{r}\cdot\mathbf{n} = l$. So:

$(\mathbf{d} \pm a\mathbf{n})\cdot\mathbf{n} = l$

$\Rightarrow \mathbf{d}\cdot\mathbf{n} \pm a = l$

$\Rightarrow \mathbf{d}\cdot\mathbf{n} = l \pm a.$

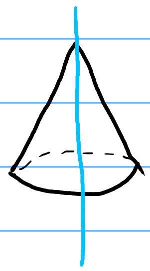
(iii)



If the radii meet at right angles, then the distance between the two centres is $\sqrt{a^2+a^2} = \sqrt{2a^2}$. But the first sphere is centred at the origin, this means $|\mathbf{d}| = \sqrt{2a^2}$

$\Rightarrow \mathbf{d}\cdot\mathbf{d} = 2a^2$

STEP III 1998 Q9



The moment of intertia of a solid disc of mass m and radius r is $\frac{1}{2}mr^2$.

Hence if the cone has density ρ, the moment of intertia for the cone is

$$\int_0^h \frac{1}{2}\cdot\underbrace{\left(\frac{ay}{h}\right)^2}_{r^2}\cdot\underbrace{\pi\rho\left(\frac{ay}{h}\right)^2 dy}_{m}$$

$$= \int_0^h \frac{1}{2}\pi\rho\frac{a^4}{h^4}y^4\,dy$$

$$= \frac{\rho\pi a^4}{2h^4}\left[\frac{1}{5}y^5\right]_0^h$$

$$= \frac{\pi\rho a^4 h}{10}$$

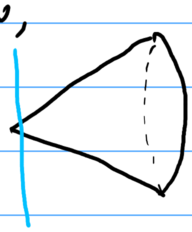
Now the mass of the cone is $m = \frac{1}{3}\pi a^2 h\rho$

$$\Rightarrow \rho = \frac{3m}{\pi a^2 h}$$

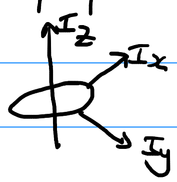
so $I = \frac{\pi a^4 h}{10}\cdot\frac{3m}{\pi a^2 h}$

$$= \frac{3ma^2}{10}$$

Now,



The perpendicular axis theorem says that $I_z = I_x + I_y$.



But for a disc, $I_x = I_y$ and $I_z = \frac{1}{2}mr^2$

$\Rightarrow I_x = I_y = \frac{1}{4}mr^2$.

Then the parallel axis theorem tells us that if the disc is distance y from the axis, $I = \frac{1}{4}mr^2 + my^2$.

So, the moment of inertia of the cone is

$$I = \int_0^h \frac{1}{4}\rho\pi \cdot \left(\frac{ay}{h}\right)^2 \cdot \left(\frac{ay}{h}\right)^2 + \rho\pi\left(\frac{ay}{h}\right)^2 \cdot y^2 \, dy$$

$$= \frac{a^2\pi\rho}{4h^2}\int_0^h \frac{a^2y^4}{h^2} + 4y^4 dy$$

$$= \frac{a^2\pi\rho}{4h^2}\left(\frac{a^2}{h^2}+4\right)\left[\frac{1}{5}y^5\right]_0^h$$

$$= \frac{a^2\pi\rho}{4h^2}\cdot\frac{h^3}{5}(a^2+4h^2)$$

$$= \frac{a^2\pi\rho h}{20}(a^2+4h^2)$$

As before, $\rho = \frac{3m}{\pi a^2 h}$, so

$$I = \frac{a^2\pi h}{20} \cdot \frac{3m}{\pi a^2 h}(a^2+4h^2)$$

$$= \frac{3m}{20}(a^2+4h^2)$$

If l is the distance from the apex to the centre of mass, then

$$I\ddot{\theta} = mgl\sin\theta$$

Now θ is small, so $\sin\theta \approx \theta$. Further, the CoM is located a distance k from the apex where

$$\frac{1}{m}\int_0^h \pi \cdot \rho \cdot \left(\frac{ay}{h}\right)^2 y \, dy$$

$$= \frac{3}{\pi a^2 h\rho} \cdot \pi\rho\frac{a^2}{h^2}\left[\frac{1}{4}y^4\right]_0^h$$

$$= \frac{3}{4}h$$

So, we have

$$\frac{\cancel{3}\cancel{m}}{20}(a^2+4h^2)\ddot{\theta} = -\cancel{m}g\theta \cdot \frac{\cancel{3}}{4}h$$

$$\Rightarrow \ddot{\theta} = \frac{-5gh}{a^2+4h^2}\theta$$

which is SHM with period $2\pi\sqrt{\frac{4h^2+a^2}{5gh}}$, as required.

STEP III 1998 Q10

Momentum: $\underline{v}_1 + \underline{v}_2 = \underline{v}_1' + \underline{v}_2'$ $\quad (*)$

Energy: $\underline{v}_1 \cdot \underline{v}_1 + \underline{v}_2 \cdot \underline{v}_2 = \underline{v}_1' \cdot \underline{v}_1' + \underline{v}_2' \cdot \underline{v}_2'$ $\quad (\dagger)$

Dotting each side of $(*)$ with itself,

$$\underline{v}_1 \cdot \underline{v}_1 + 2\underline{v}_1 \cdot \underline{v}_2 + \underline{v}_2 \cdot \underline{v}_2 = \underline{v}_1' \cdot \underline{v}_1' + 2\underline{v}_1' \cdot \underline{v}_2' + \underline{v}_2' \cdot \underline{v}_2'$$

Then subtracting $(\dagger)$, we obtain $\underline{v}_1 \cdot \underline{v}_2 = \underline{v}_1' \cdot \underline{v}_2'$ $\quad (\ddagger)$

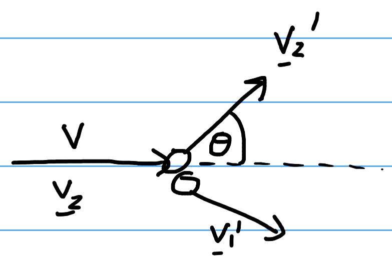
Now, the square of the initial relative speed is

$$\begin{aligned}(\underline{v}_1 - \underline{v}_2)\cdot(\underline{v}_1 - \underline{v}_2) &= \underline{v}_1 \cdot \underline{v}_1 - 2\underline{v}_1 \cdot \underline{v}_2 + \underline{v}_2 \cdot \underline{v}_2 \\ &= \underline{v}_1 \cdot \underline{v}_1 + \underline{v}_2 \cdot \underline{v}_2 - 2\underline{v}_1 \cdot \underline{v}_2 \\ &= \underline{v}_1' \cdot \underline{v}_1' + \underline{v}_2' \cdot \underline{v}_2' - 2\underline{v}_1' \cdot \underline{v}_2' \quad \text{by } (\dagger) \text{ and } (\ddagger) \\ &= (\underline{v}_1' - \underline{v}_2')\cdot(\underline{v}_1' - \underline{v}_2'), \text{ the square of the final relative speed.}\end{aligned}$$

Since speed > 0, the inital and final relative speed are the same.

wlog now let $\underline{v}_1 = 0$. Then $\underline{v}_1 \cdot \underline{v}_2 = 0 \Rightarrow \underline{v}_1' \cdot \underline{v}_2' = 0$

$\Rightarrow \underline{v}_1'$ & $\underline{v}_2'$ are perpendicular (as they are non-zero)



As $\underline{v}_1 = 0$, we have $\underline{v}_2 = \underline{v}_1' + \underline{v}_2'$

$\Rightarrow \underline{v}_1' = \underline{v}_2 - \underline{v}_2'$ $\quad (\ddagger)$

Also, $\underline{v}_1' \cdot \underline{v}_1' + \underline{v}_2' \cdot \underline{v}_2' = (1-k)V^2$.

Substituting in $(\ddagger)$, $(\underline{v}_2 - \underline{v}_2')\cdot(\underline{v}_2 - \underline{v}_2') + \underline{v}_2' \cdot \underline{v}_2' = (1-k)V^2$

$$\Rightarrow \underline{v_2}\cdot\underline{v_2} - 2\underline{v_2}\cdot\underline{v_2}' + \underline{v_2}'\cdot\underline{v_2}' + \underline{v_2}'\cdot\underline{v_2}' = (1-k)V^2$$

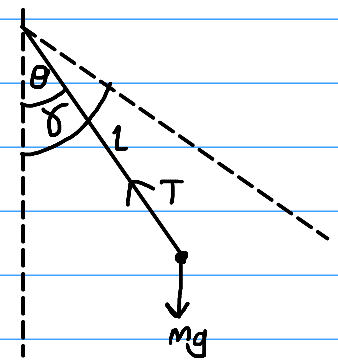
Letting $|\underline{v_2}| = x$, we have

$$V^2 - 2xV\cos\theta + 2x^2 = (1-k)V^2$$
$$\Rightarrow 2x^2 - 2V\cos\theta\, x + kV^2 = 0$$

Since this has solutions, the discrimant is positive, so

$$4V^2\cos^2\theta - 8kV^2 \geqslant 0$$
$$\Rightarrow k \leq \tfrac{1}{2}\cos^2\theta$$
$$\Rightarrow k \leq \tfrac{1}{2} \text{ as } \cos^2\theta \leq 1.$$

STEP III 1998 Q11



$GPE = mg(l\cos\gamma - l\cos\theta)$

$KE = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2$

So by conservation of energy,

$$\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 = -mgl(\cos\gamma - \cos\theta)$$

$$\rightarrow \frac{1}{2}l\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta - \cos\gamma)$$

So, $\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\gamma}$

$$= \sqrt{\frac{2g}{l}}\left(1 - 2\sin^2\frac{\theta}{2} - 1 + 2\sin^2\frac{\gamma}{2}\right)^{1/2}$$

$$= \sqrt{\frac{2g}{l}}\sqrt{2(\sin^2\frac{\gamma}{2} - \sin^2\frac{\theta}{2})}$$

$$= 2\sqrt{\frac{g}{l}}\sqrt{\sin^2\frac{\gamma}{2} - \sin^2\frac{\theta}{2}}$$

Now, the time taken to go from $\theta = 0$ to $\theta = \gamma$ is

$$\int_0^\gamma \frac{dt}{d\theta}\,d\theta$$

$$= \frac{1}{2}\sqrt{\frac{l}{g}}\int_0^{\gamma} \frac{1}{\sqrt{\sin^2\frac{\gamma}{2} - \sin^2\frac{\theta}{2}}}\, d\theta$$

The period is 4 times this, so $P =$

$$P = 2\sqrt{\frac{l}{g}}\int_0^{\gamma} \frac{1}{\sqrt{\sin^2\frac{\gamma}{2} - \sin^2\frac{\theta}{2}}}\, d\theta, \quad \text{as required.}$$

Now setting $\sin\frac{\theta}{2} = \sin\frac{\gamma}{2}\sin\phi$, so

$$\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = \sin\frac{\gamma}{2}\cos\phi\, d\theta$$, we have

$$P = 2\sqrt{\frac{l}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{\sin^2\frac{\gamma}{2}(1-\sin^2\phi)}} \cdot \frac{2\sin\frac{\gamma}{2}\cos\phi}{\cos\theta/2}\, d\phi$$

$$= 2\sqrt{\frac{l}{g}}\int_0^{\pi/2} \frac{1}{\sin\frac{\gamma}{2}\cos\phi} \cdot \frac{2\sin\frac{\gamma}{2}\cos\phi}{\cos\theta/2}\, d\phi$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} \frac{1}{\cos\theta/2}\, d\phi$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} \left(1 - \sin^2\frac{\gamma}{2}\sin^2\phi\right)^{-1/2} d\phi$$

$$\approx 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} \left(1 + \frac{1}{2}\sin^2\frac{\gamma}{2}\sin^2\phi\right) d\phi \qquad \text{(binomial expansion)}$$

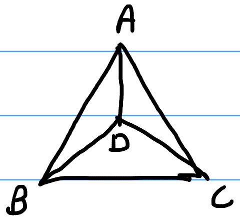
$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} 1 + \frac{1}{2}\sin^2\frac{\gamma}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 2\phi\right) d\phi$$

$$= 4\sqrt{\frac{l}{g}}\left[\phi + \frac{1}{4}\sin^2\frac{\gamma}{2}\cdot\phi - \frac{1}{8}\sin^2\frac{\gamma}{2}\sin 2\phi\right]_0^{\pi/2}$$

$$= 4\sqrt{\frac{l}{g}}\left(\pi/2 + \pi/8\sin^2\frac{\gamma}{2}\right)$$

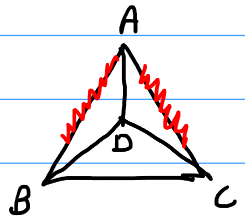
$$\approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{4}\left(\frac{\gamma}{2}\right)^2\right) = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\gamma^2}{16}\right) \qquad \text{(small angle approximation)}$$

STEP III 1998 Q12



If 0, 1, or 2 roads are blocked, all villages are still accessible. If three roads are blocked, one village could have all three roads blocked, with probability $p^3(1-p)^3$. There are 4 villages, so the total probability is $4p^3(1-p)^3$.

What if no village has all 3 roads blocked? Well, each road has 2 'ends', so 6 blocked ends in total. There are 4 villages, so (at least) one village has 2 roads out of it blocked, wlog make this A.



Then unless AD is also blocked, all villages are still accessible. So - with 3 blocked roads, all villages are accessible unless one village has all 3 roads blocked, which happens with probability $4p^3(1-p)^3$.

Suppose 4 roads are blocked, so two roads are open. wlog AB is open. No one other road can connect AB to both C & D, so not all villages are accessible. This happens with probability $\binom{6}{4}p^4(1-p)^2 = 15p^4(1-p)^2$.

Similarly if 5 or 6 roads are blocked not all villages are accessible, which occurs with probabilities $6p^5(1-p)$ and p^6 respectively.

$$\text{So } P = 1-\left(4p^3(1-p)^3 + 15p^4(1-p)^2 + 6p^5(1-p) + p^6\right)$$
$$= 1-p^3\left(4(1-3p+3p^2-p^3) + 15p(1-2p+p^2) + 6p^2(1-p) + p^3\right)$$
$$= 1-p^3\left(4-12p+12p^2-4p^3+15p-30p^2+15p^3+6p^2-6p^3+p^3\right)$$
$$= 1-p^3(6p^3-12p^2+3p+4), \text{ as required.}$$

STEP III 1998 Q13

$P(k \text{ heads out of } n) = \binom{n}{k} p^k (1-p)^{n-k}$
$= \binom{n}{k}\left(\frac{1}{2}\right)^n$

$L(n) = \binom{n}{1}\left(\frac{1}{2}\right)^n$
$= n \cdot \left(\frac{1}{2}\right)^n$

Note $\frac{L(n+1)}{L(n)} = \frac{(n+1)\left(\frac{1}{2}\right)^{n+1}}{n\left(\frac{1}{2}\right)^n} = \frac{1}{2} \cdot \frac{n+1}{n}$
$= \frac{1}{2}\left(1 + \frac{1}{n}\right) < 1$ if $1 + \frac{1}{n} < 2$
$\Rightarrow \frac{1}{n} < 1$
$\Rightarrow n > 1$

So, for $n > 1$, $L(n+1) < L(n)$. So only need to consider $n = 1$ & $n = 2$
$L(1) = \frac{1}{2}$
$L(2) = 2 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{2}$
So 1 & 2 are both MLE

Now, k heads are observed, so $L(n) = \binom{n}{k}\left(\frac{1}{2}\right)^n$
$= \frac{n!}{k!(n-k)!}\left(\frac{1}{2}\right)^n$

So $\frac{L(n+1)}{L(n)} = \frac{(n+1)!}{k!(n+1-k)!} \cdot \left(\frac{1}{2}\right)^{n+1} \times \frac{k!(n-k)!}{n!} \times \frac{1}{\left(\frac{1}{2}\right)^n}$

$= \frac{1}{2} \cdot \frac{n+1}{n+1-k} < 1$
$\Rightarrow n+1 < 2(n+1-k)$
$\Rightarrow n+1 < 2n+2-2k$
$\Rightarrow n > 2k-1$

So $L(n+1) < L(n)$ for $n > 2k-1$ and similarly $L(n+1) > L(n)$ for $n < 2k-1$.

For $n = 2k-1$, $\frac{L(n+1)}{L(n)} = 1 \Rightarrow L(2k-1) = L(2k)$.

So $n = (2k-1)$ and $n = 2k$ are both MLE.

Now $L(n) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$, so

$$\frac{L(n+1)}{L(n)} = \frac{(n+1)!}{k!(n+1-k)!} p^k (1-p)^{n+1-k} \cdot \frac{k!(n-k)!}{n!} \times \frac{1}{p^k (1-p)^{n-k}}$$

$$= \frac{n+1}{n+1-k}(1-p)$$

If $L(n) = L(n+1)$, then $\frac{n+1}{n+1-k}(1-p) = 1$

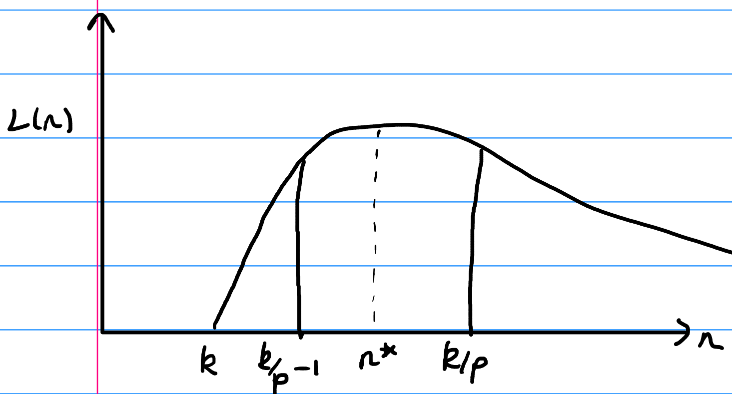
$\Rightarrow (n+1)(1-p) = n+1-k$

$\Rightarrow n - np + 1 - p = n+1-k$

$\Rightarrow np = k - p$

$\Rightarrow n = k/p - 1$

Now $L(n)$ is increasing up to some value n^*, then decreasing.



So $k/p - 1 < n^* < k/p$.

If $k/p \in \mathbb{Z}$, then $n = k/p - 1$ & $n = k/p$ are both MLE.

If $k/p \notin \mathbb{Z}$, then the MLE is the unique integer $\in (k/p - 1, k/p)$.

So the MLE is unique if and only if k/p is not an integer.

STEP III 1998 Q14

$Z_1 = \frac{n+1}{n} \text{Max}(X_1, \_, X_n)$

$P(\text{Max}(X_1, \_, X_n)) < x$

$= P(X_1 < x \cap X_2 < x \cap \_ \cap X_n < x)$

$= P(X_1 < x)^n$

$= \left(\frac{x}{N}\right)^n$

So, the PDF of $\text{Max}(X_1, \_, X_n)$ is $\frac{d}{dx}\left(\frac{x}{N}\right)^n = \frac{n}{N}\left(\frac{x}{N}\right)^{n-1}$

So $EZ_1 = E\frac{n+1}{n}\text{Max}(X_1, \_, X_n)$

$= \frac{n+1}{n}\int_0^N x \cdot \frac{n}{N}\left(\frac{x}{N}\right)^{n-1} dx$

$= \frac{n+1}{N}\left[\frac{1}{N^{n-1}} \cdot \frac{x^{n+1}}{n+1}\right]_0^N$

$= \frac{n+1}{N^n} \cdot \frac{N^{n+1}}{n+1}$

$= N$

$EZ_2 = \frac{2}{n}\sum_{i=1}^{n} EX_i$

$= \frac{2}{n} \cdot n \cdot \frac{N}{2}$

$= N$

So $EZ_1 = EZ_2 = N$

Now $EZ_1^2 = \left(\frac{n+1}{n}\right)^2 \int_0^N x^2 \cdot \frac{n}{N} \cdot \left(\frac{x}{N}\right)^{n-1} dx$

$= \frac{(n+1)^2}{n \times N^n}\left[\frac{x^{n+2}}{n+2}\right]_0^N$

$$= \frac{(n+1)^2}{n(n+2)} \cdot N^2$$

So $\text{Var} Z_1 = E Z_1^2 - (E Z_1)^2$

$$= \frac{(n+1)^2}{n(n+2)} \cdot N^2 - N^2$$

$$= N^2 \left(\frac{(n+1)^2}{n(n+2)} - \frac{n(n+2)}{n(n+2)} \right)$$

$$= N^2 \left(\frac{n^2 + 2n + 1 - n^2 - 2n}{n(n+2)} \right)$$

$$= \frac{N^2}{n(n+2)}$$

$$\text{Var} Z_2 = \frac{4}{n^2} \sum_{i=1}^{n} \text{Var} X_i$$

$$= \frac{4}{n^2} \cdot n \cdot \frac{1}{12} N^2$$

$$= \frac{N^2}{3n}$$

Now $\text{Var} Z_1 < \text{Var} Z_2 \iff \frac{N^2}{n(n+2)} < \frac{N^2}{3n}$

$$\iff 3n < n(n+2)$$

$$\iff n^2 - n > 0$$

$$\iff n > 1 \text{ (or } n < 0)$$

So $\text{Var} Z_1 \leq \text{Var} Z_2$, with equality only for $n = 1$.

So we prefer Z_1, as both are unbiased, but Z_1 has a lower variance.