# STEP II 1998 Q1

$n^3 = (n+3)^3 - (n-3)^3$.

Suppose $n$ is odd, then $(n+3)$ and $(n-3)$ are even $\Rightarrow (n+3)^3$ and $(n-3)^3$ are even, so $(n+3)^3 - (n-3)^3$ is even, so $n^3$ is even, so $n$ is even, a contradiction. So $n$ must be even.

Expanding, we have

$n^3 = n^3 + 9n^2 + 27n + 27 - n^3 + 9n^2 - 27n + 27$

$\Rightarrow n^3 = 18n^2 + 54$

$\Rightarrow n^2(n-18) = 54$

$\Rightarrow n^2$ is a factor of 54 (and $n^2 \leq 54 \Rightarrow |n| < 8$).

So $|n| = 2, 4,$ or $6$ (as $n$ even).

But $2^2, 4^2$, and $6^2$ do not divide 54. So no $n$ exists which satisfies $(*)$.

Now $n^3 = (n+6)^3 - (n-6)^3$.

If $n$ is odd, then $(n+6)$ and $(n-6)$ are odd $\Rightarrow (n+6)^3$ and $(n-6)^3$ are odd $\Rightarrow (n+6)^3 - (n-6)^3$ even $\Rightarrow n^3$ even $\Rightarrow n$ even $\#$. So $n$ is even.

Expanding,

$n^3 = n^3 + 18n^2 + 108n + 216 - n^3 + 18n^2 - 108n + 216$

$\Rightarrow n^3 = 36n^2 + 216$

$\Rightarrow n^2(n-36) = 216 \quad (\dagger)$

$\Rightarrow n^2$ is a factor of 216 $\Rightarrow |n| = 2, 4, 6, 8, 10, 12, 14$. With all of these values of $n$, the LHS of $(\dagger)$ is negative, and 216 is positive.

So no values of $n$ satisfy $(**)$.

STEP II 1998 Q2

$$\left(1-\frac{2}{100}\right)^{1/2} = 1+\left(\frac{1}{2}\right)\left(\frac{-2}{100}\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-2}{100}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-2}{100}\right)^3}{6}$$

$$= 1 - \frac{1}{100} - \frac{1}{2}\cdot\frac{1}{10{,}000} - \frac{1}{2}\,\frac{1}{1{,}000{,}000}$$

$$= 1 - 0.01 - 0.00005 - 0.0000005$$

$$= 0.9899495$$

Now, $\left(1-\frac{2}{100}\right)^{1/2} = \sqrt{\frac{98}{100}} = \frac{7}{10}\sqrt{2}$

So $\sqrt{2} \approx \frac{10}{7} \times 0.9899495$

$$\begin{array}{r} 1.4142135\ldots \\ = 7 \overline{)\,9.8994950} \end{array}$$

$\approx 1.414214$

Now $\left(1+\frac{N}{125}\right)^{1/3} = \left(\frac{N+125}{125}\right)^{1/3}$

$$= \frac{(N+125)^{1/3}}{5}$$

Taking $N=3$, this is $\frac{\sqrt[3]{128}}{5} = \frac{4\sqrt[3]{2}}{5}$

$$\left(1+\frac{3}{125}\right)^{1/3} \approx 1 + \frac{1}{3}\cdot\frac{3}{125} + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{3}{125}\right)^2}{2} + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{3}{125}\right)^3}{6}$$

$$= 1 + \frac{1}{125} - \frac{1}{125^2} + \frac{5}{3\times125^3}$$

$$= 1 + 0.008 - 0.000064 + \frac{5}{3} \times 0.000000512$$

$$= 1.007936 + 0.00000085333\ldots$$

So $\sqrt[3]{2} \approx \frac{5}{4} \times 1.0079369$

$$= \frac{1}{4} \times 5.0396845$$

$$\begin{array}{r} 1.2599211 \\ = 4 \overline{)\,5.0396845} \end{array}$$

$\approx 1.259921$

STEP II 1998 Q3

$$S_N = \sum_{n=1}^{N} \frac{2n-1}{n(n+1)(n+2)}$$

Now $\dfrac{2n-1}{n(n+1)(n+2)} \equiv \dfrac{A}{n} + \dfrac{B}{n+1} + \dfrac{C}{n+2}$

$\Leftrightarrow \quad 2n-1 \equiv A(n+1)(n+2) + Bn(n+2) + Cn(n+1)$

$n=0 \Rightarrow A = -1/2$

$n=-1 \Rightarrow B = 3$

$n=-2 \Rightarrow C = -5/2$

$$\text{So } S_N = \sum_{n=1}^{N} \frac{-1}{2n} + \frac{3}{n+1} - \frac{5}{2(n+2)}$$

$$= \frac{1}{2}\sum_{n=1}^{N} -\frac{1}{n} + \frac{6}{n+1} - \frac{5}{(n+2)}$$

$$= \frac{1}{2}\left(\sum_{n=1}^{N} -\frac{1}{n} + \sum_{n=2}^{N+1}\frac{6}{n} - \sum_{n=3}^{N+2}\frac{5}{n}\right)$$

$$= \frac{1}{2}\left(-1 - \frac{1}{2} + \frac{6}{2} + \sum_{n=3}^{N}\left(-\frac{1}{n} + \frac{6}{n} - \frac{5}{n}\right) + \frac{6}{N+1} - \frac{5}{N+1} + \frac{5}{N+2}\right)$$

$$= \frac{1}{2}\left(-1 - \frac{1}{2} + 3 + 0 + \frac{1}{N+1} - \frac{5}{N+2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2}\right), \text{ as required.}$$

As $N \to \infty$, $S_N \to 3/4$

$$\frac{a_n}{a_{n-1}} = \frac{(n-1)(2n-1)}{(n+2)(2n-3)}$$

$$\frac{a_n}{a_{n-2}} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} = \frac{(n-1)(2n-1)}{(n+2)\cancel{(2n-3)}} \times \frac{(n-2)\cancel{(2n-3)}}{(n+1)(2n-5)}$$

$$\frac{a_n}{a_1} = \frac{\cancel{(n-1)}(2n-1)}{(n+2)\cancel{(2n-3)}} \times \frac{\cancel{(n-2)}\cancel{(2n-3)}}{(n+1)\cancel{(2n-5)}} \times \frac{\cancel{(n-3)}\cancel{(2n-5)}}{n\cancel{(2n-7)}} \times \frac{\cancel{(n-4)}\cancel{(2n-7)}}{\cancel{(n-1)}\cancel{(2n-9)}} \times \dots \times \frac{2\times\cancel{5}}{\cancel{5}\times\cancel{3}} \times \frac{1\times\cancel{3}}{\cancel{4}\times 1}$$

$$= \frac{(2n-1)\times 3\times 2\times 1}{(n+2)(n+1)(n)}$$

$$= \frac{6(2n-1)}{n(n+1)(n+2)}$$

So $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a_n}{a_1} \cdot a_1$

$$= \sum_{n=1}^{\infty} \frac{6(2n-1)}{n(n+1)(n+2)} \cdot \frac{2}{9}$$

$$= \frac{4}{3} \cdot \sum_{n=1}^{\infty} \frac{(2n-1)}{n(n+1)(n+2)}$$

$$= \frac{4}{3} \cdot \frac{3}{4}$$

$$= 1$$

STEP II 1998 Q4

$$I_n - I_{n-1} = \int_0^{\pi} (\pi/2 - x)\operatorname{cosec}(x/2)\left[\sin(nx + x/2) - \sin(nx - x/2)\right] dx$$

$$= \int_0^{\pi} (\pi/2 - x)\operatorname{cosec}(x/2)\left(\cancel{\sin nx \cos x/2} + \cos nx \sin x/2 - \cancel{\sin nx \cos x/2} + \cos nx \sin x/2\right) dx$$

$$= 2\int_0^{\pi} (\pi/2 - x)\cancel{\operatorname{cosec} x/2}\, \cancel{\sin x/2} \cos nx \, dx$$

$$= 2\int_0^{\pi} (\pi/2 - x)\cos nx \, dx$$

$u \quad \pi/2 - x \qquad v' \quad \cos nx$

$u' \quad -1 \qquad v \quad \frac{1}{n}\sin nx$

$$= \underbrace{2\left[(\pi/2 - x)\tfrac{1}{n}\sin nx\right]_0^{\pi}}_{=0} + \frac{2}{n}\int_0^{\pi} \sin nx \, dx$$

$$= \frac{-2}{n^2}\left[\cos nx\right]_0^{\pi}$$

$$= \frac{-2}{n^2}\left((-1)^n - 1\right)$$

$$= \frac{2}{n^2}\left(1 - (-1)^n\right)$$

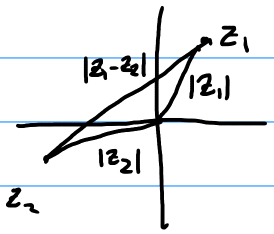
Now $I_0 = \int_0^{\pi} (\pi/2 - x)\, dx = 0$

So $I_n = (I_n - I_{n-1}) + (I_{n-1} - I_{n-2}) + \_\_\_\_ + (I_1 - I_0) + I_0$

$$= \sum_{i=1}^{n} \frac{2}{i^2}\left(1 - (-1)^i\right)$$

STEP II 1998 Q5

The modulus of a complex number $z = x + iy$ for $x, y \in \mathbb{R}$ is $|z| = \sqrt{x^2+y^2}$. $|z_1 - z_2|$ is the distance between $z_1$ and $z_2$ in an Argand diagram.




We have $|z_1 - z_2| \le |z_1| + |z_2|$ (the triangle inequality)

Claim: $|z_1 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$

For $n=2$, proved above.

Assume true for $n=k$.

For $n=k+1$, $|z_1 + \cdots + z_k + z_{k+1}| \le |z_1 + \cdots + z_k| + |z_{k+1}|$ ($n=2$ case)

$\le |z_1| + \cdots + |z_k| + |z_{k+1}|$ ($n=k$ case)

True for $n=2$, and if true for $n=k$ then true for $n=k+1$, so true for all $n \in \mathbb{N}$ ($n=1$ is trivially true).

We have $|a_i| \le 3$. Assume $|z| \le \frac{1}{4}$

$$|a_1 z + \cdots + a_n z^n| \le |a_1 z| + \cdots + |a_n z^n|$$
$$= |a_1||z| + \cdots + |a_n||z^n|$$
$$\le 3(|z| + |z|^2 + \cdots + |z|^n)$$
$$\le 3\left(\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{4}\right)^2 + \cdots \left(\tfrac{1}{4}\right)^n\right)$$
$$= 3\,\frac{\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)}{3/4}$$
$$= 1 - \left(\tfrac{1}{4}\right)^n$$
$$< 1$$

So $a_1 z + a_2 z^2 + \cdots + a_n z^n = 1$ has no solutions for $|a_i| \le 3$, $|z| \le \frac{1}{4}$.

STEP II 1998 Q6

$C_1$: $x = \theta + \sin\theta \qquad y = 1 + \cos\theta$

$C_2$: $x = \theta - \sin\theta \qquad y = -(1 + \cos\theta)$

For $C_1$, $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{d\theta}{dx} = \frac{-\sin\theta}{1+\cos\theta}$

For $C_2$, $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$

At $\theta = \pi/2$, $m_{c_1} = \frac{-\sin\pi/2}{1+\cos\pi/2} = -1$

$m_{c_2} = \frac{\sin\pi/2}{1-\cos\pi/2} = 1$

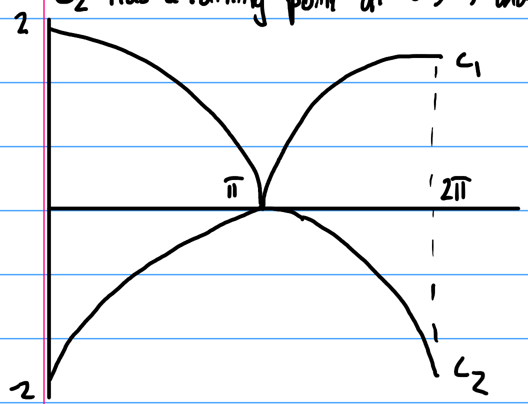
At $\theta = 3\pi/2$, $m_{c_1} = 1$

$m_{c_2} = -1$

$C_1$: $\frac{dy}{dx} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi \qquad (0,2), (\pi,0), (2\pi,2)$

but $\theta = \pi$ gives $\frac{dy}{dx}$ undefined, so curve is not differentiable there (but still continuous)

$C_2$ has a turning point at $(\pi, 0)$ and passes through $(0,-2)$ and $(2\pi,-2)$




Now the normal to $C_1$ has equation $y - (1+\cos\theta) = \frac{1+\cos\theta}{\sin\theta}(x - (\theta + \sin\theta))$. Setting $x = (\theta - \sin\theta)$, $y = -(1+\cos\theta)$, we have $-2(1+\cos\theta) = \frac{1+\cos\theta}{\sin\theta}(-2\sin\theta) \Rightarrow -2(1+\cos\theta) = -2(1+\cos\theta)$ which is true.

So this line intersects $C_2$ at the point with the parameter $\theta$.

Further, the gradient of $C_2$ at this point is $\frac{\sin\theta}{1-\cos\theta} = \frac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} = \frac{1+\cos\theta}{\sin\theta}$, the gradient of the line. So the line is tangent to $C_2$ at this point.

STEP II 1998 Q7

(i) $f(0) = 0$

$f'(x) = \sec^2 x - 1 \geqslant 0$ for $0 < x < \pi/2$

So $f$ is increasing on $[0, \pi/2]$, and $f(0) = 0$, so $f(x) > 0$ on $(0, \pi/2)$.

(ii) $g'(x) = 2\sin x - \sin x - x\cos x$

$= \sin x - x\cos x$

$g''(x) = \cos x - \cos x + x\sin x$

$= x\sin x > 0$ for $x \in (0, \pi/2)$.

$g'(0) = 0$ and $g''(x) > 0$, so $g'(x) > 0$ for $x \in (0, \pi/2)$.

$g(0) = 2 - 2 - 0 = 0$, and $g'(x) > 0$, so $g(x) > 0$ for $x \in (0, \pi/2)$.

(iii) $h(0) = 0$

$h'(x) = 2 + \cos 2x - 2x\sin 2x - 3\cos 2x$

$= 2 - 2\cos 2x - 2x\sin 2x$

By (ii) this is $> 0$ on $(0, \pi/4)$

$h(0) = 0$ and $h'(x) > 0$, so $h(x) > 0$ on $(0, \pi/4)$.

But $h(x) = x(2 + \cos 2x) - 3\sin x\cos x$

$= x(\sin^2 x + \cos^2 x + 1 + 2\cos^2 x - 1) - 3\sin x\cos x$

$= x(\sin^2 x + 3\cos^2 x) - 3\sin x\cos x > 0$ on $(0, \pi/4)$ as shown above.

(iv) $F(x) = \dfrac{x(\cos x)^{1/3}}{\sin x}$ $\quad F'(x) = \dfrac{\sin x\left((\cos x)^{1/3} - \frac{1}{3}x\sin x(\cos x)^{-2/3}\right) - x(\cos x)^{4/3}}{\sin^2 x}$

So $\dfrac{F'(x)}{F(x)} = \dfrac{\cancel{(\cos x)^{1/3}}\left[\sin x - \frac{1}{3}x\sin x(\cos x)^{-1} - x\cos x\right]}{\sin^{\cancel{2}} x} \cdot \dfrac{\cancel{\sin x}}{x\cancel{(\cos x)^{1/3}}}$

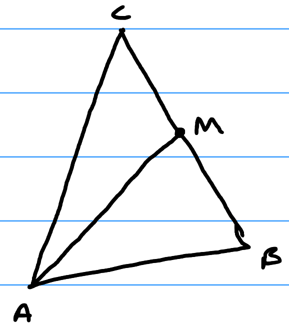
$= \dfrac{\sin x - \frac{1}{3}x\sin x(\cos x)^{-1} - x\cos x}{x\sin x}$

$$= \frac{\overbrace{3\sin x\cos x - x\sin^2 x - 3x\cos^2 x}^{<0 \text{ by (iii)}}}{\underbrace{3x\sin x\cos x}_{>0 \text{ on } (0,\pi/4)}}$$

$$< 0$$

So $\frac{F'(x)}{F(x)} < 0$ on $(0, \pi/4)$. But $F(x) = \frac{x(\cos x)^{1/3}}{\sin x} > 0$ on $(0,\pi/4)$ clearly, so we must have $F'(x) < 0$ on $(0, \pi/4)$.

STEP II 1998 Q8




We have $\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$

$= -\underline{a} + \underline{b} + \frac{1}{2}(-\underline{b} + \underline{c})$

$= -\underline{a} + \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}$

So the line through A&M has equation $\underline{r} = \underline{a} + \lambda(-\underline{a} + \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c})$,

Similarly, the line through B and the midpoint of AC has equation $\underline{r} = \underline{b} + \mu(\frac{1}{2}\underline{a} - \underline{b} + \underline{c})$, and the line through C and the midpoint of AB has equation $\underline{r} = \underline{c} + \delta(\frac{1}{2}\underline{a} + \frac{1}{2}\underline{b} - \underline{c})$.

Setting $\lambda = \mu = \delta = 2/3$ gives $\underline{r} = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$ for all three lines. This is the point R.

The line through the midpoint of BC and parallel to PA has equation

$$\underline{r} = \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c} + \lambda(\underline{p} - \underline{a})$$

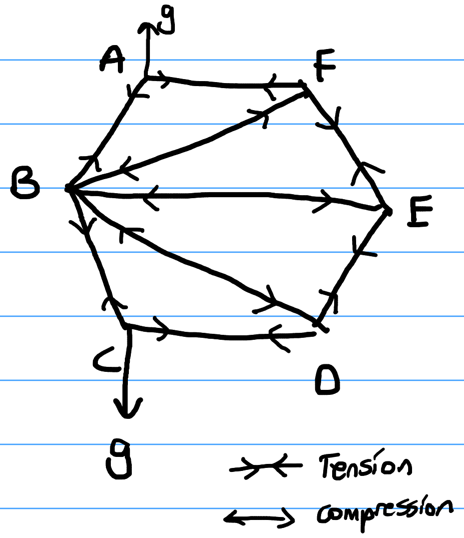
and the similar lines through the midpoints of the other sides are

$$\underline{r} = \frac{1}{2}\underline{a} + \frac{1}{2}\underline{c} + \mu(\underline{p} - \underline{b})$$
$$\underline{r} = \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b} + \delta(\underline{p} - \underline{c})$$

Setting $\lambda = \mu = \delta = -\frac{1}{2}$, $Q = \frac{1}{2}(\underline{a} + \underline{b} + \underline{c} - \underline{p})$ lies on all three lines. Then PQ has equation $\underline{r} = \underline{p} + \lambda(\frac{1}{2}(\underline{a} + \underline{b} + \underline{c} - \underline{p}) - \underline{p}) = \underline{p} + \lambda(\frac{1}{2}\underline{a} + \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c} - \frac{3}{2}\underline{p})$. This meets ABC when the $\underline{p}$ component is zero, so $1 - \frac{3}{2}\lambda = 0 \Rightarrow \lambda = 2/3$.

Then $r = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) = R$, as required.

# STEP II 1998 Q9




By N2L, the force at A is vertically upwards, of magnitude $g$. Then AB is in tension, so AF is in tension. Similarly BC and CD are in tension.

If BF is in tension, then FE must be acting rightwards and upwards, which is impossible. So BF is in compression and FE in tension. Similarly BD is in compression and DE in tension. Then by resolving horizontally at E, BE must be in compression.

Now, resolving vertically at A, the force in AB is $\frac{g}{\cos 30} = \frac{2g}{\sqrt{3}}$. Then resolving horizontally the force in AF is $AB \sin 30 = \frac{g}{\sqrt{3}}$

Then at F, resolving horizontally, $AF = BF\cos 30 + EF \cos 60$

$$g/\sqrt{3} = \sqrt{3}/2\, BF + \tfrac{1}{2} EF \quad (*)$$

resolving vertically, $BF \sin 30 = EF \sin 60$

$$\tfrac{1}{2} BF = \frac{\sqrt{3}}{2} EF$$

$$\Rightarrow \frac{\sqrt{3}}{2} BF = \frac{3}{2} EF$$

Substituting into $(*)$, $g/\sqrt{3} = \frac{3}{2} EF + \frac{1}{2} EF$

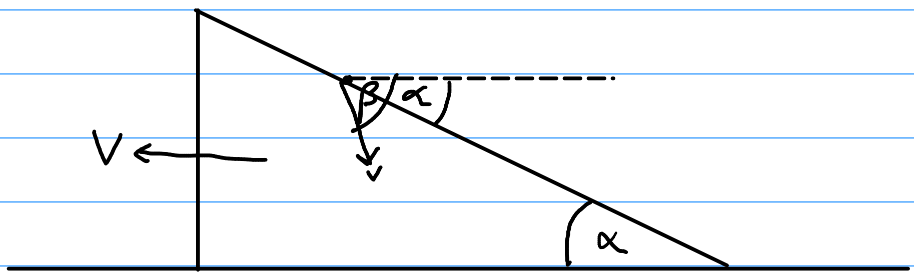
$$\Rightarrow EF = g/2\sqrt{3}$$

$$\Rightarrow BF = g/2$$

By symmetry, the force in DE = force in EF. So resolving horizontally at E,

$$BF = 2 \times EF \cos 60$$
$$= 2 \times \frac{g}{2\sqrt{3}} \times \frac{1}{2}$$
$$= \frac{g}{2\sqrt{3}}$$
$$= \frac{g\sqrt{3}}{6}$$

STEP II 1998 Q9




(i) Given that the particle stays in contact with the plane, the component of velocity perpendicular to the plane is zero. So, $v\sin(\beta-\alpha) = V\sin\alpha$.

(ii) Considering horizontal conservation of momentum,

$$MV - mv\cos\beta = 0 \qquad \left(\Rightarrow V = \frac{m}{M}v\cos\beta \ (*)\right)$$

$\Rightarrow MV\sin\alpha - mv\sin\alpha\cos\beta = 0$

$\Rightarrow Mv\sin(\beta-\alpha) - mv\sin\alpha\cos\beta = 0 \qquad$ (By (i))

$\Rightarrow v(M\sin\beta\cos\alpha - M\sin\alpha\cos\beta - m\sin\alpha\cos\beta) = 0$

$\Rightarrow M\sin\beta\cos\alpha = (M+m)\sin\alpha\cos\beta$

$\Rightarrow M\tan\beta = (M+m)\tan\alpha$

$\Rightarrow \tan\beta = (1+m/M)\tan\alpha$

(iii) By conservation of energy,

$\frac{1}{2}MV^2 + \frac{1}{2}mv^2 = mgy$

$\Rightarrow 2gy = \frac{M}{m}V^2 + v^2$

$= \frac{M}{m}\cdot v^2\frac{m^2\cos^2\beta}{M^2} + v^2 \qquad$ (by (*))

$= v^2\left(\frac{m}{M}\cos^2\beta + 1\right)$

$= \frac{v^2}{M}(m\cos^2\beta + M)$, as required.

Differentiating this w.r.t. $t$,

$$2g\dot{y} = 2v\dot{v}\left(\frac{M + m\cos^2\beta}{M}\right)$$

But $\dot{y} = v\sin\beta$, so $v = \frac{\dot{y}}{\sin\beta}$, $\dot{v} = \frac{\ddot{y}}{\sin\beta}$

So $$g\cancel{\dot{y}} = \frac{\cancel{\dot{y}}\,\ddot{y}}{\sin^2\beta}\left(\frac{M + m\cos^2\beta}{M}\right)$$

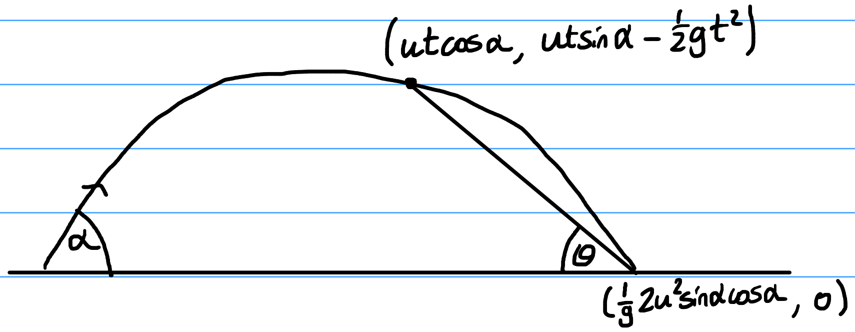
$$\Rightarrow \ddot{y} = \frac{Mg\sin^2\beta}{M + m\cos^2\beta}$$

Then integrating twice, and noting that $y(0) = \dot{y}(0) = 0$,

$$\dot{y} = \frac{Mg\sin^2\beta\, t}{M + m\cos^2\beta}$$

$$\Rightarrow y = \frac{gMt^2\sin^2\beta}{2(M + m\cos^2\beta)}$$

STEP II 1998 Q11

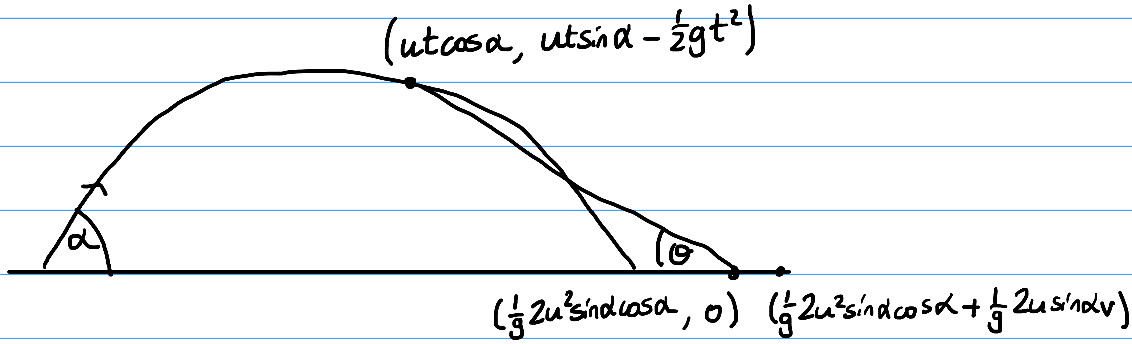



We have $\tan\theta = \dfrac{ut\sin\alpha - \frac{1}{2}gt^2}{\frac{1}{g}2u^2\sin\alpha\cos\alpha - ut\cos\alpha}$

$$= \frac{t(\cancel{u\sin\alpha - \frac{1}{2}g})}{\frac{2u}{g}\cos\alpha(\cancel{u\sin\alpha - \frac{1}{2}g})}$$

$$= \frac{gt}{2u\cos\alpha}$$

So $\frac{d}{dt}(\tan\theta) = \dfrac{g}{2u\cos\alpha}$ which is constant.




$$\tan\theta = \frac{ut\sin\alpha - \frac{1}{2}gt^2}{\frac{1}{g}2u^2\sin\alpha\cos\alpha + \frac{1}{g}2uv\sin\alpha - vt - ut\cos\alpha}$$

$$= \frac{t(u\sin\alpha - \frac{1}{2}g)}{\frac{2u\cos\alpha}{g}(u\sin\alpha - \frac{1}{2}g) + \frac{2v}{g}(u\sin\alpha - \frac{1}{2}gt)}$$

$$= \frac{gt}{2(u\cos\alpha + v)}$$

So $\frac{d}{dt}\tan\theta = \dfrac{g}{2(u\cos\alpha + v)}$ which is constant.

STEP II 1998 Q12

$$P(\text{sufferer} \mid AL) = \frac{P(AL \mid \text{sufferer})P(\text{sufferer})}{P(AL)}$$

$$= \frac{\frac{9}{10} \times \frac{1}{100}}{\frac{9}{10} \times \frac{1}{100} + \frac{1}{11} \times \frac{99}{100}}$$

$$= \frac{9/1000}{9/1000 + 9/100}$$

$$= 9/99$$

$$= 1/11$$

$$P(AL \,\&\, STEP \,\&\, \text{sufferer}) = P(AL \,\&\, STEP \,\&\, \text{non-sufferer})$$

$$\Rightarrow \quad \frac{1}{\cancel{100}} \times \frac{9}{10} \times \frac{9}{10} = \frac{99}{\cancel{100}} \times \frac{1}{11} \times p$$

$$\Rightarrow \quad \frac{81}{100} = 9p$$

$$\Rightarrow \quad p = \frac{9}{100}$$

STEP II 1998 Q13

$$\int f(x)\,dx = \int \lambda e^{-\lambda x}\,dx$$
$$= -e^{-\lambda x} + c$$

But $F(0)=0 \Rightarrow c=1$

so $F(x) = 1 - e^{-\lambda x}$

Then $P(X > s+t \mid X > t) = \dfrac{P(X > s+t \cap X > t)}{P(X > t)}$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$
$$= e^{-\lambda s}$$
$$= P(X > s)$$

$P(\text{both still being served}) = P(X > t)^2$
$$= e^{-2\lambda t}$$

By memorylessness as established,

$P(\text{I am still being served when other customer finishes}) = \frac{1}{2}$,

as the opposite outcome is equally likely

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$$E\frac{B}{1+x} = B\sum_{x=0}^{N}\frac{1}{1+x}\binom{N}{x}p^x(1-p)^{N-x}$$

$$= B\sum_{x=0}^{N}\frac{1}{1+x}\,\frac{N!}{x!(N-x)!}p^x(1-p)^{N-x}$$

$$= B\sum_{x=0}^{N}\frac{N!}{(x+1)!(N-x)!}p^x(1-p)^{N-x}$$

$$= B\sum_{x=0}^{N}\frac{1}{p(N+1)}\,\frac{(N+1)!}{(x+1)!(N+1-(x+1))!}p^{x+1}(1-p)^{N+1-(x+1)}$$

$$= \frac{B}{p(N+1)}\sum_{x=1}^{N+1}\binom{N+1}{x}p^x(1-p)^{N-x}$$

$$= \frac{B}{p(N+1)}\left(1-(1-p)^{N+1}\right)$$

So accept the new salary if

$$A < \frac{B}{p(N+1)}\left(1-(1-p)^{N+1}\right)$$

$$\Rightarrow A(N+1)p < B\left(1-(1-p)^{N+1}\right)$$

X can be approximated by a Poisson random variable if N is large and p is small, with $\mu = np$. In this case,

$$E\frac{B}{1+x} = B\sum_{x=0}^{\infty}\frac{1}{1+x}\cdot\frac{\mu^x e^{-\mu}}{x!}$$

$$= B\sum_{x=0}^{\infty}\frac{\mu^x e^{-\mu}}{(x+1)!}$$

$$= \frac{B}{\mu}\sum_{x=0}^{\infty}\frac{\mu^{x+1}e^{-\mu}}{(x+1)!}$$

$$= \frac{B}{\mu}\sum_{x=1}^{\infty}\frac{\mu^x e^{-\mu}}{x!}$$

$$= \frac{B}{\mu}(\mu - e^{-\mu})$$

$$= B\left(1 - \frac{e^{-\mu}}{\mu}\right) \quad \text{where } \mu = np.$$