STEP I 1998 Q1

There are $\binom{10}{2} = 45$ possible pairs of digits.

If neither of the digits are 0, then there are $2^5 = 32$ possibilities. But 2 of these have only one digit, so there are 30 possibilities. There are $\binom{9}{2} = 36$ pairs of non-zero digits.

If one of the digits is 0, the first digit of the new number must be the non-zero number (to be $\geq 10{,}000$), then there is free choice for the other 4 digits, so $2^4 = 16$ possibilities. One of these has only one digit, so there are 15 possibilities. There are 9 such pairs including a zero.

Also, we need to include 100,000 as well.

So, the total is $36 \times 30 + 15 \times 9 + 1$

$= 1216$

STEP I 1998 Q2

Set $t = (x^2+1)^{1/2} + x$, so $\frac{dt}{dx} = \frac{x}{\sqrt{x^2+1}} + 1$

$$\int_0^\infty f\left((x^2+1)^{1/2} + x\right) dx = \int_1^\infty f(t) \cdot \frac{1}{\frac{x}{\sqrt{x^2+1}} + 1}\, dt$$

$$= \int_1^\infty f(t) \frac{\sqrt{x^2+1}}{x + \sqrt{x^2+1}}\, dt$$

Now $1 + t^{-2}$

$$= 1 + \frac{1}{(x+\sqrt{x^2+1})^2}$$

$$= \frac{x^2 + 2x\sqrt{x^2+1} + x^2 + 1 + 1}{(x+\sqrt{x^2+1})^2}$$

$$= 2\,\frac{x\sqrt{x^2+1} + (x^2+1)}{(x+\sqrt{x^2+1})^2}$$

$$= 2\sqrt{x^2+1}\;\frac{x+\sqrt{x^2+1}}{(x+\sqrt{x^2+1})^2}$$

$$= \frac{2\sqrt{x^2+1}}{(x+\sqrt{x^2+1})}$$

So our integral becomes $\frac{1}{2}\int_1^\infty f(t)(1+t^{-2})\,dt$, as required.

Taking $f(x) = x^{-3}$, $\int_0^\infty \left((x^2+1)^{1/2} + x\right)^{-3} dx$

$$= \tfrac{1}{2}\int_1^\infty t^{-3}(1+t^{-2})\,dt$$

$$= \tfrac{1}{2}\int_1^\infty t^{-3} + t^{-5}\,dt$$

$$= \tfrac{1}{2}\left[-t^{-2}/2 - t^{-4}/4\right]_1^\infty$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{4}\right)$$

$= 3/8$, as required.

STEP I 1998 Q3

(i) This is true. Taking natural log of both sides,

$$\ln(a^{\ln b}) = \ln(b^{\ln a})$$
$$\Leftrightarrow \ln b \ln a = \ln a \ln b$$

which is true. So the statement is true.

(ii) This is false. Take $\theta = 0$.

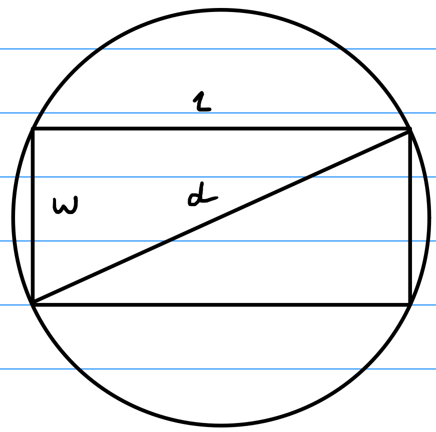
Then $\cos(\sin(0)) = \cos(0) = 1$

$\sin(\cos(0)) = \sin(1) \neq 1$

(iii) This is false. Suppose such a polynomial exists. Then as $x \longrightarrow \infty$, $|P(x)| \longrightarrow \infty$. In particular, there exists some α such that $P(\alpha) = 2$. But $|\cos\theta| \leq 1$ for all θ, so $|P(\alpha) - \cos\alpha| \geq 1$, a contradiction.

(iv) This is true. If $y = x^4 + 3 + x^{-4}$, then $\frac{dy}{dx} = 4x^3 - 4x^{-5}$. Setting this equal to zero, we have $x^8 = 1 \Rightarrow x = 1$ (not -1 as $x > 0$). $\frac{d^2y}{dx^2} = 12x^2 + \frac{20}{x^6} > 0$, so this is a minimum point. $y(1) = 1 + 3 + 1 = 5$. The function is continuous on $(0, \infty)$, and $(1, 5)$ is the only stationary point and is a minimum. So the function is greater than or equal to 5 for $x > 0$.

STEP I 1998 Q4



Suppose the rectangle has length l and width w. Then we want to maximise $l+w$. Now
$l^2+w^2=d^2 \Rightarrow w=\sqrt{d^2-l^2}$

so maximise $P=l+\sqrt{d^2-l^2}$

$$\frac{dP}{dl}=1-\frac{l}{\sqrt{d^2-l^2}}=0$$

$$\Rightarrow l^2=d^2-l^2$$

$$\Rightarrow l=d/\sqrt{2}$$

$$\Rightarrow w=d/\sqrt{2}$$, so a square.

For $n=2$, want to maximise $l^2+w^2=l^2+d^2-l^2$
$=d^2$ which is constant.

For $n=3$, want to maximise $l^3+w^3=l^3+(d^2-l^2)^{3/2}$
Differentiating, we obtain

$$3l^2-3l\sqrt{d^2-l^2}=0$$

$$\Rightarrow l(l-\sqrt{d^2-l^2})=0$$

$$\Rightarrow l=0 \text{ or } l-\sqrt{d^2-l^2}=0$$

$$\Rightarrow l=0 \text{ or } l=d/\sqrt{2}$$

When $l=0$ (or when $l=d$) the sum of third powers is $2d^3$
When $l=w=d/\sqrt{2}$, the sum of third powers is $2\left(\frac{\sqrt{2}}{2}\right)^3 d^3=\frac{\sqrt{2}}{2}d^3<2d^3$
So the maximum of the sum of third powers is $2d^3$, when the rectangle has zero area.

STEP I 1998 Q5

(i) To find the next vertex, we need to translate by $-(4+6i)$, rotate by $-\pi/3$, then translate by $(4+6i)$. So,

$$B = [(10+2i)-(4+6i)]e^{-i\pi/3} + (4+6i)$$
$$= (6-4i)(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i) + 4+6i$$
$$= 3-3\sqrt{3}\,i - 2i - 2\sqrt{3} + 4+6i$$
$$= (7-2\sqrt{3}) + (4-3\sqrt{3})i$$

(ii) $$\frac{a}{1+i} + \frac{b}{1+2i} + \frac{c}{1+3i} = \frac{a(1-i)}{2} + \frac{b(1-2i)}{5} + \frac{c(1-3i)}{10}$$

Setting the imaginary part equal to zero,

$$\frac{1}{10}(-5a-4b-3c) = 0 \implies 5a+4b+3c=0$$

STEP I 1998 Q6

$a_2 = \frac{1}{2}(1+\cos x) = \cos^2\frac{x}{2}$

$a_3 = \frac{1}{2}\left(\cos^2\frac{x}{2} + \cos\frac{x}{2}\right)$

$= \frac{1}{2}\cos\frac{x}{2}\left(\cos\frac{x}{2}+1\right)$

$= \cos\frac{x}{2}\cos^2\frac{x}{4}$

$b_2 = \sqrt{\frac{1}{2}(1+\cos x)} = \cos\frac{x}{2}$

$b_3 = \sqrt{\cos\frac{x}{2}\cos^2\frac{x}{4}\cos\frac{x}{2}}$

$= \cos\frac{x}{2}\cos\frac{x}{4}$

We hypothesise $a_n = \cos\frac{x}{2^{n-1}} \cdot \prod_{i=1}^{n-1}\cos\frac{x}{2^i}$ and $b_n = \prod_{i=1}^{n-1}\cos\frac{x}{2^i}$

Clearly true for $n=2$ and $n=3$. Assume true for $n=k$. Then

$a_{k+1} = \frac{1}{2}\left(\cos\frac{x}{2^{k-1}}\prod_{i=1}^{k-1}\cos\frac{x}{2^i} + \prod_{i=1}^{n-1}\cos\frac{x}{2^i}\right)$

$= \prod_{i=1}^{k-1}\cos\frac{x}{2^i} \times \frac{1}{2}\left(\cos\frac{x}{2^i}+1\right)$

$= \prod_{i=1}^{k-1}\cos\frac{x}{2^i} \times \cos^2\frac{x}{2^k}$

$= \cos\frac{x}{2^k}\cdot\prod_{i=1}^{k}\cos\frac{x}{2^i}$, as required.

$b_{k+1} = \left(\cos\frac{x}{2^k}\prod_{i=1}^{k}\cos\frac{x}{2^i}\cdot\prod_{i=1}^{k-1}\cos\frac{x}{2^i}\right)^{1/2}$

$= \left(\prod_{i=1}^{k}\cos\frac{x}{2^i}\cdot\prod_{i=1}^{k}\cos\frac{x}{2^i}\right)^{1/2}$

$= \prod_{i=1}^{k}\cos\frac{x}{2^i}$, as required.

STEP I 1998 Q7

Set $r = 1 + \frac{p}{100}$

After 1 year, we have ra

After 2 years, $r^2a + ra$

After 3 years, $r^3a + r^2a + ra$

After m years $\sum_{i=1}^{m} a \cdot r^i = \frac{ar(r^m - 1)}{r-1}$

Then l more years with no deposits $= r^l \cdot \frac{ar(r^m-1)}{r-1}$

$= £a\frac{r^{l+1}(r^m-1)}{r-1}$, as required.

Let $N = a\frac{r^{l+1}(r^m-1)}{r-1}$.

For withdrawals, after 1 year $r(N-b) = rN - b$

2 years $r(r(N-b)-b) = r^2N - r^2b - rb$

n years $= r^nN - b\sum_{i=1}^{n} r$

$= r^nN - \frac{r(r^n-1)}{r-1}b$

So, we need

$$r^n \cdot a\frac{r^{l+1}(r^m-1)}{r-1} - \frac{r(r^n-1)}{r-1}b = 0$$

$$\Rightarrow ar^{n+l}(r^m-1) - b(r^n-1) = 0$$

$$\Rightarrow \frac{a}{b} = \frac{r^n-1}{r^{n+l}(r^m-1)}$$

STEP I 1998 Q8

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = -k$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{dV}{dr}\right) = -kr$$

$$\Rightarrow r\frac{dV}{dr} = -\frac{1}{2}kr^2 + A$$

$$\Rightarrow \frac{dV}{dr} = \frac{-1}{2}kr + \frac{A}{r}$$

$$\Rightarrow V = \frac{-1}{4}kr^2 + A\ln r + B$$

Now $|\ln r| \longrightarrow \infty$ as $r \longrightarrow 0$, so require $A=0$.

$V=0$ when $r=a \Rightarrow 0 = \frac{-1}{4}ka^2 + B$

$\Rightarrow B = \frac{1}{4}ka^2$

So $V = \frac{1}{4}k(a^2 - r^2)$.

Now $F = 2\pi\int_0^a rV\,dr = \frac{\pi k}{2}\int_0^a a^2r - r^3\,dr$

$$= \frac{\pi k}{2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^a$$

$$= \frac{\pi k}{2}\left(\frac{a^4}{4}\right)$$

$$= \frac{\pi k a^4}{8}$$

STEP I 1998 Q9

Both pendulums are the same length so have the same period. They collide at the lowes point, so leave this point at the same time, so return to this point at the same time, for the next collision. This process repeats.

First collision:

Before: $\xrightarrow{u}$ $\quad 0$

After: $\xrightarrow{v_1}$ $\quad \xrightarrow{v_2}$

CoM: $u = v_1 + v_2$

e: $eu = v_2 - v_1$

$\Rightarrow v_2 = \frac{1}{2}u(1+e)$

$v_1 = \frac{1}{2}u(1-e)$

Second collision:

Before: $\xleftarrow{\frac{1}{2}u(1-e)}$ $\quad \xleftarrow{\frac{1}{2}u(1+e)}$

After: $\xleftarrow{v_1}$ $\quad \xleftarrow{v_2}$

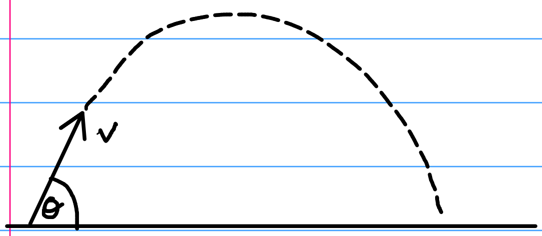
CoM: $u = v_1 + v_2$

e: $e^2u = v_1 - v_2$

$\Rightarrow v_1 = \frac{1}{2}u(1+e^2)$

$v_2 = \frac{1}{2}u(1-e^2)$, as required.

STEP I 1998 Q10



Horizontally, $\frac{v\cos\theta}{r} = t$

$\Rightarrow \cos\theta = \frac{r}{vt}$

Vertically,

s	0
u	$v\sin\theta$
v	$-v\sin\theta$
a	$-g$
t	t

$t = \frac{v-u}{a} \Rightarrow \frac{2v\sin\theta}{g} = t$

$\Rightarrow \sin\theta = \frac{tg}{2v}$

Now $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \frac{t^2g^2}{4v^2} + \frac{r^2}{v^2t^2} = 1$

$\Rightarrow t^4\left(\frac{g^2}{4}\right) - t^2v^2 + r^2 = 0$

$\Rightarrow t^2 = \frac{v^2 \pm \sqrt{v^4 - g^2r^2}}{\frac{1}{2}g^2}$

$\Rightarrow \frac{1}{2}g^2t^2 = v^2 \pm \sqrt{v^4 - g^2r^2}$, as required.

Vertically, $s = ut + \frac{1}{2}at^2$

$= vt\sin\theta - \frac{1}{2}gt^2$

$v = u + at$

$0 = v\sin\theta - gt$

$\Rightarrow \sin\theta = \frac{gt}{v}$

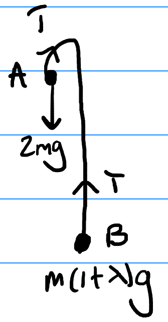
So $s = vt \cdot \frac{gt}{v} - \frac{1}{2}gt^2$

$= gt^2 - \frac{1}{2}gt^2$

$= \frac{1}{2}gt^2$

$= \frac{1}{g}\left(v^2 \pm \sqrt{v^4 - g^2r^2}\right)$

STEP I 1998 Q11



For A: $2mg - T = 2ma \Rightarrow T = 2mg - 2ma$

For B: $T - mg(1+\lambda) = m(1+\lambda)a$

so $2mg - 2ma - mg(1+\lambda) = m(1+\lambda)a$

$\Rightarrow g(1-\lambda) = a(3+\lambda)$

$\Rightarrow a = \frac{g(1-\lambda)}{3+\lambda}$

Now $s = ut + \frac{1}{2}at^2$ and $u=0$ so $t = \sqrt{\frac{2l(3+\lambda)}{g(1-\lambda)}}$

For the lowering of bucket B, bucket A has no mass and so B experience freefall under gravity. So $l = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2l}{g}}$

So total time is $\sqrt{\frac{2l(3+\lambda)}{g(1-\lambda)}} + \sqrt{\frac{2l}{g}}$

To maximise ore, need to maximise ore per time, which is

$$\lambda \div \left(\sqrt{\frac{2l(3+\lambda)}{g(1-\lambda)}} + \sqrt{\frac{2l}{g}}\right)$$

$$= \lambda \div \left(\frac{\sqrt{2l(3+\lambda)} + \sqrt{2l(1-\lambda)}}{\sqrt{g(1-\lambda)}}\right)$$

$$= \sqrt{\frac{g}{2l}}\,\frac{\lambda(1-\lambda)^{1/2}}{(1-\lambda)^{1/2} + (3+\lambda)^{1/2}}$$

$\sqrt{\frac{g}{2l}}$ is a constant, so we want to maximise $f(\lambda) = \frac{\lambda(1-\lambda)^{1/2}}{(1-\lambda)^{1/2} + (3+\lambda)^{1/2}}$

$f(0) = 0$ and $f(1) = 0$, and f is continuous and positive on $(0,1)$, so we require λ s.t. $f'(\lambda) = 0$.

STEP I 1998 Q12

X and Y are not independent. $P(X=10) \neq 0$ but $P(X=10 \mid Y=11)=0$.

Suppose $X=k$. Then $Y+Z=20-k$ and there are $(20-k+1)$ ways for this to happen ($Y=0,1,2,\ldots,20-k$). So in total there are

$$\sum_{k=0}^{20}(20-k+1) = \sum_{k=0}^{20} k+1$$

$$= \sum_{k=1}^{21} k$$

$$= \frac{21 \times 22}{2}$$

$= 231$ possibilities.

Now X is divisible by 5, so $X = 0, 5, 10, 15, 20$. There are 21, 16, 11, 6, 1 possibilities for Y & Z in each case, for a total of 55. So, $P(X$ is divisible by $5)$ is $\frac{55}{231} = \frac{5}{21}$

$P(XYZ$ is divisible by $5) = P(5 \mid X) + P((5 \mid Y \text{ or } 5 \mid Z) \cap 5 \nmid X)$.

Now $P(5 \mid X) = \frac{5}{21}$

We calculate $P((5 \mid Y \text{ or } 5 \mid Z) \cap 5 \nmid X)$ by considering different values of X. For $1 \leq X \leq 4$, either Y or Z can be 0, 5, 10, or 15, for a total of $4 \times 8 = 32$ possibilities. For $6 \leq X \leq 9$, Y or Z can be 0, 5, or 10, for a total of $4 \times 6 = 24$ possibilities. Similarly for $11 \leq X \leq 14$ there are 16 possibilities, and for $16 \leq X \leq 19$ there are 8 possibilities, for a total of 80 possibilities.
So $P((5 \mid Y \text{ or } 5 \mid Z) \cap 5 \nmid X) = \frac{80}{231}$.
(Note we cannot have $5 \nmid X$ but $5 \mid Y$ and $5 \mid Z$ because $X+Y+Z=20$)

Hence the total probability is $\frac{5}{21} + \frac{80}{231} = \frac{45}{77}$

STEP I 1998 Q13

$P(n^{th} \text{ draw is red} \mid \text{exactly one red in first } n)$

$$= \frac{P(n^{th} \text{ draw is red} \cap \text{exactly one red in first } n)}{P(\text{exactly one red in first } n)}$$

$$= \frac{\left(\frac{b}{b+r}\right)^{n-1}\left(\frac{r}{b+r}\right)}{\left(\frac{r}{b+r}\right)\left(\frac{b}{b+r-1}\right)^{n-1} + \left(\frac{b}{b+r}\right)\left(\frac{r}{b+r}\right)\left(\frac{b}{b+r-1}\right)^{n-2} + \ldots + \left(\frac{b}{b+r}\right)^{n-1}\left(\frac{r}{b+r}\right)}$$

$$= \frac{(b+r)^{-n}}{(b+r)^{-1}(b+r-1)^{-(n-1)} + (b+r)^{2}(b+r-1)^{-(n-2)} + \ldots + (b+r)^{-n}}$$

$$= \frac{1}{\left(\frac{b+r}{b+r-1}\right)^{n-1} + \left(\frac{b+r}{b+r-1}\right)^{n-2} + \ldots + \left(\frac{b+r}{b+r-1}\right)^{0}}$$

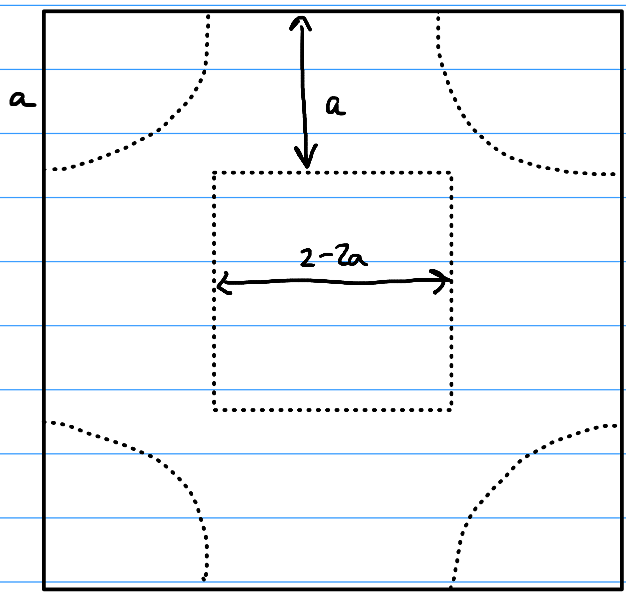
$$= \frac{1}{\dfrac{\left(\frac{b+r}{b+r-1}\right)^{n} - 1}{\left(\frac{b+r}{b+r-1}\right) - 1}}$$

$$= \frac{\frac{b+r}{b+r-1} - 1}{\left(\frac{b+r}{b+r-1}\right)^{n} - 1}$$

$$= \frac{\frac{1}{b+r-1}}{\left(\frac{b+r}{b+r-1}\right)^{n} - 1}$$

$$= \frac{(b+r-1)^{n-1}}{(b+r)^{n} - (b+r-1)^{n-1}}, \text{ as required.}$$

STEP I 1998 Q14



The centre of the coin falls uniformly at random in the square. Tax stays the same if the centre of the coin is inside the centre square, or inside one of the quarter circles.

This occurs with probability $\frac{(2-2a)^2}{4} + \frac{\pi a^2}{4}$

$$= \tfrac{1}{4}\left(4 - 8a + 4a^2 + \pi a^2\right)$$

To raise the most tax, we want to minimise this probability. Differentiating and setting to zero, we have

$$\tfrac{1}{4}(-8 + 8a + 2\pi a) = 0$$

$$\Rightarrow \quad a(\pi + 4) = 4$$

$$\Rightarrow \quad a = \frac{4}{\pi+4}$$

$$= \left(\frac{\pi+4}{4}\right)^{-1}$$

$$= \left(1 + \pi/4\right)^{-1}$$

The second derivative is $2 + \pi/2 > 0$ so this stationary point is a minimum.

Now, the number of times the tax doubles is $B(n, p)$, where p is the probability of tax doubling in a given year.

So $p = 1 - \frac{1}{4}(4 - 8a + 4a^2 + \pi a^2)$ where $a = \frac{4}{4+\pi}$

$$= 1 - \frac{1}{4}\left(4 - \frac{32}{4+\pi} + (4+\pi)\left(\frac{4}{4+\pi}\right)^2\right)$$
$$= 1 - \frac{1}{4}\left(4 - \frac{32}{4+\pi} + \frac{16}{4+\pi}\right)$$
$$= 1 - \left(1 - \frac{4}{4+\pi}\right)$$
$$= \frac{4}{4+\pi}$$

So, the expected value of the tax after n years is

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^k \cdot 2^k$$

value of tax if doubled k times

$$= \sum_{k=0}^{n} \binom{n}{k} (2p)^k (1-p)^k$$

$$= (2p + 1 - p)^n$$
$$= (1+p)^n$$
$$= \left(1 + \frac{4}{4+\pi}\right)^n$$

$$= \left(\frac{8+\pi}{4+\pi}\right)^n$$, as required.