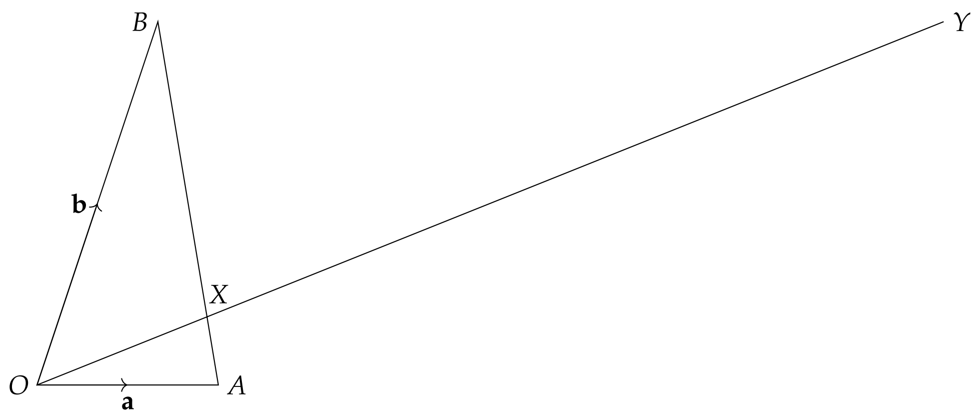
Vectors

1.

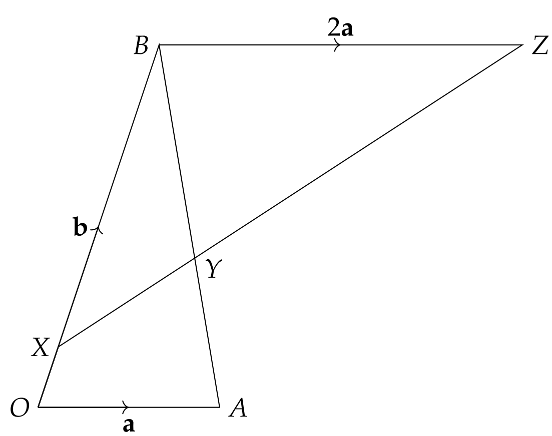


In the diagram,

- $\overrightarrow{OA} = \mathbf{a}$
- $\overrightarrow{OB} = \mathbf{b}$
- $AX : XB = 1 : 4$
- $\overrightarrow{OY} = 5 \times \overrightarrow{OX}$

Show that $\overrightarrow{OA}$ and $\overrightarrow{BY}$ are parallel.

2.

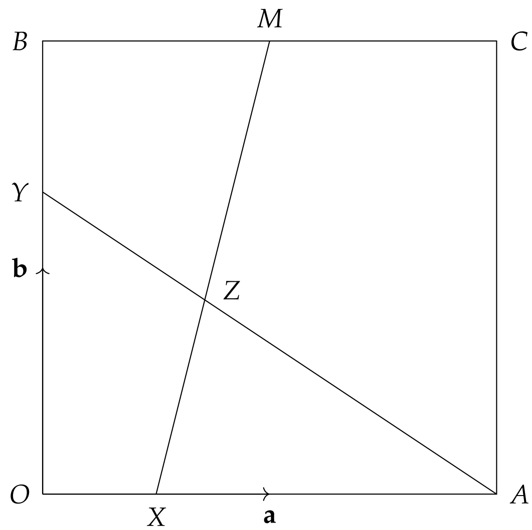


In the diagram,

- $\overrightarrow{OA} = \mathbf{a}$
- $\overrightarrow{OB} = \mathbf{b}$
- $\overrightarrow{BZ} = 2\mathbf{a}$
- $OX : XB = 1 : 5$

Find the ratio $BY : BA$.

3.

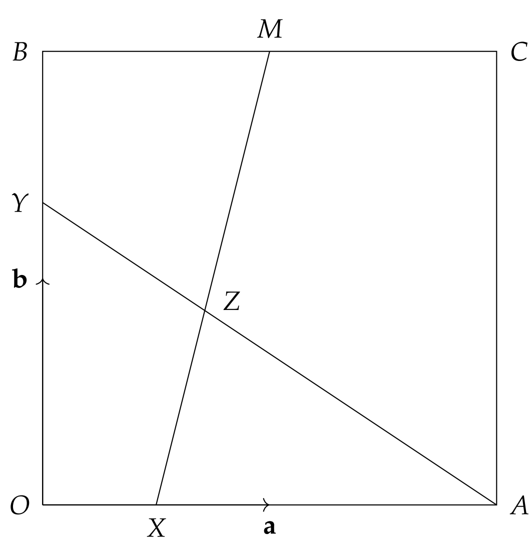


In the diagram,

- $ABCD$ is a square.
- $\overrightarrow{OA} = \mathbf{a}$
- $\overrightarrow{OB} = \mathbf{b}$
- M is the midpoint of BC
- $OY : YB = 2 : 1$
- $OX : XA = 1 : 3$

Find the vector $\overrightarrow{OZ}$.

4.



In the diagram,

- $ABCD$ is a square.

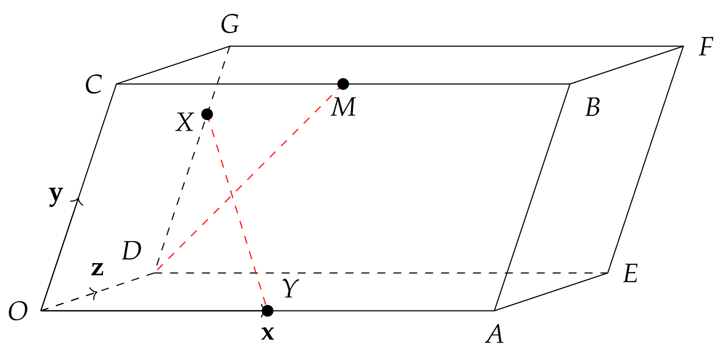
- $\overrightarrow{OA} = \mathbf{a}$
- $\overrightarrow{OB} = \mathbf{b}$
- M is the midpoint of BC
- $\overrightarrow{OX} = m \times \overrightarrow{OA}$
- $\overrightarrow{OY} = n \times \overrightarrow{OB}$

Find the vector $\overrightarrow{OZ}$, giving your answer in terms of m and n.

5. A **median** of a triangle is a line segment passing through a vertex and the midpoint of the opposite side. For example, in triangle ABC one of the medians is the line segment between A and the midpoint of BC.

Using a vector method, show that the three medians of a triangle meet at a single point.

6.



In the diagram,

- $OABCDEFG$ is a parallelepiped.
- $\overrightarrow{OA} = \mathbf{x}$
- $\overrightarrow{OC} = \mathbf{y}$
- $\overrightarrow{OD} = \mathbf{z}$
- M is the midpoint of BC
- Y is the midpoint of OA
- $\overrightarrow{DX} = k \times \overrightarrow{DG}$

Show that, for every $0 < k < 1$, the lines XY and MD intersect. If the point of intersection is Z, find the vector $\overrightarrow{OZ}$, giving your answer in terms of k.

1. $\overrightarrow{OX} = \mathbf{a} + \frac{1}{5}(-\mathbf{a}+\mathbf{b}) = \frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}$. So, $\overrightarrow{OY} = 5 \times \overrightarrow{OX} = 4\mathbf{a} + \mathbf{b}$. Then $\overrightarrow{BY} = \overrightarrow{BO} + \overrightarrow{OY} = -\mathbf{b} + 4\mathbf{a} + \mathbf{b} = 4\mathbf{a} = 4 \times \overrightarrow{OA}$. So $\overrightarrow{OA}$ and $\overrightarrow{BY}$ are parallel.
2. $\overrightarrow{BY} = \lambda\overrightarrow{BA} = \lambda(-\mathbf{b}+\mathbf{a})$.
Also, $\overrightarrow{BY} = \overrightarrow{BX} + \mu\overrightarrow{XZ} = -\frac{5}{6}\mathbf{b} + \mu\left(\frac{5}{6}\mathbf{b} + 2\mathbf{a}\right)$.
Equating these, $\lambda = 2\mu$ and $-\lambda = -\frac{5}{6} + \frac{5}{6}\mu$, giving us $\lambda = \frac{10}{17}$. So $BY : BA = 10 : 7$.
3. $\overrightarrow{OZ} = \overrightarrow{OX} + \lambda\overrightarrow{XZ} = \frac{1}{4}\mathbf{a} + \lambda\left(-\frac{1}{4}\mathbf{a} + \mathbf{b} + \frac{1}{2}\mathbf{a}\right) = \frac{1}{4}\mathbf{a} + \lambda\left(\frac{1}{4}\mathbf{a} + \mathbf{b}\right)$.
Also, $\overrightarrow{OZ} = \overrightarrow{OY} + \mu\overrightarrow{YA} = \frac{2}{3}\mathbf{b} + \mu\left(-\frac{2}{3}\mathbf{b} + \mathbf{a}\right)$.
Equating these, $\frac{1}{4} - \frac{1}{4}\lambda = \mu$ and $\lambda = \frac{2}{3} - \frac{2}{3}\mu$, giving us $\lambda = \frac{3}{7}$, and so $\overrightarrow{OZ} = \frac{1}{4}\mathbf{a} + \frac{3}{7}\left(\frac{1}{4}\mathbf{a} + \mathbf{b}\right) = \frac{5}{14}\mathbf{a} + \frac{3}{7}\mathbf{b}$.
4. Using a similar method to before, we obtain $\overrightarrow{OZ} = \mathbf{a}(m - \lambda m + 0.5\lambda) + \lambda\mathbf{b}$ and also $\overrightarrow{OZ} = \mu\mathbf{a} + (n - \mu n)\mathbf{b}$. Equating and solving simultaneously, we get

$$\mu = \frac{n - 2mn + 2m}{n - 2nm + 2}$$

and so

$$\overrightarrow{OZ} = n\mathbf{b} + \frac{n - 2mn + 2m}{n - 2nm + 2}(-n\mathbf{b} + \mathbf{a}) = \frac{n - 2mn + 2m}{n - 2nm + 2}\mathbf{a} + \frac{n(2 - 2m)}{n - 2nm + 2}\mathbf{b}$$

5. Suppose the triangle has vertices OAB with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Define the midpoint of the side opposite angle X as M_X. Then $\overrightarrow{OM_O} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\overrightarrow{AM_A} = -\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\overrightarrow{BM_B} = -\mathbf{b} + \frac{1}{2}A$.
Suppose the lines AM_A and BM_B meet at the point X. Then $\overrightarrow{OX} = \overrightarrow{OA} + \lambda\overrightarrow{AM_A} = \mathbf{a} + \lambda(-\mathbf{a} + \frac{1}{2}\mathbf{b})$. But also $\overrightarrow{OX} = \overrightarrow{OB} + \mu\overrightarrow{BM_B} = \mathbf{b} + \mu(-\mathbf{b} + \frac{1}{2}\mathbf{a})$. Equating these, we obtain $\lambda = \mu = \frac{2}{3}$. Then $\overrightarrow{OX} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} = \frac{2}{3}\overrightarrow{OM_O}$, so X lies on $OM_{,O}$, as required.
6. $\overrightarrow{OZ} = \frac{k}{2(1+k)}\mathbf{x} + \frac{1}{1+k}\mathbf{y} + \frac{1}{1+k}\mathbf{z}$.