STEP III 1997 Q1

(i) $$(x^2+5x+4)e^x = (x^2+5x+4)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+ \text{—}\right)$$

$$= 4+(4+5)x+\left(\frac{4}{2!}+\frac{5}{1!}+1\right)x^2+\left(\frac{4}{3!}+\frac{5}{2!}+\frac{1}{1!}\right)x^3+\dots$$
$$+\left(\frac{4}{n!}+\frac{5}{(n-1)!}+\frac{1}{(n-2)!}\right)x^n$$

$$= 4+(4+5)x+\sum_{n=2}^{\infty}\left(\frac{4}{n!}+\frac{5}{(n-1)!}+\frac{1}{(n-2)!}\right)x^n$$

$$= 4+(4+5)x+\sum_{n=2}^{\infty}\frac{1}{n!}\left(4+5n+n(n-1)\right)x^n$$

$$= 4+(4+5)x+\sum_{n=2}^{\infty}\frac{1}{n!}(n^2+4n+4)x^n$$

$$= 4+(4+5)x+\sum_{n=2}^{\infty}\frac{(n+2)^2}{n!}x^n$$

$$= 4+\frac{3^2}{1!}x+\frac{4^2}{2!}x^2+\frac{5^2}{3!}x^3+\dots$$

Setting $x=1$, we have
$$10e = 4+\frac{3^2}{1!}+\frac{4^2}{2!}+\frac{5^2}{3!}+\dots$$, as required.

(ii) The numerators are $(n+1)^2 = n^2+2n+1$
$$= n(n-1)+3n+1$$

So $$\frac{(n+1)^2}{n!} = \frac{1}{n!}\left(n(n-1)+3n+1\right)$$
$$= \frac{1}{(n-2)!}+\frac{3}{(n-1)!}+\frac{1}{n!}$$

So use $(x^2+3x+1)e^x$

$$(x^2+3x+1)e^x = (x^2+3x+1)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+ \text{—}\right)$$

$$= 1+(1+3)x+\sum_{n=2}^{\infty}\left(\frac{1}{(n-2)!}+\frac{3}{(n-1)!}+\frac{1}{n!}\right)x^n$$

$$= 1+4x+\sum_{n=2}^{\infty}\frac{(n+1)^2}{n!}x^n$$

$$= 1 + \frac{2^2}{1!}x + \frac{3^2}{2!}x^2 + \frac{4^2}{3!}x^3 + \dots$$

Setting $x=1$, we have

$$5e = 1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \dots \text{, as required.}$$

(iii) $(n+1)^3 = n^3 + 3n^2 + 3n + 1$

$= n(n-1)(n-2) + 6n(n-1) + 7n + 1$

So use $x^3 + 6x^2 + 7x + 1$

$$(x^3 + 6x^2 + 7x + 1)e^x = (x^3 + 6x^2 + 7x + 1)(1 + x + x^2/2! + \dots)$$

$$= 1 + 8x + \frac{27}{2}x^2 + \sum_{n=3}^{\infty}\left(\frac{1}{(n-3)!} + \frac{6}{(n-2)!} + \frac{7}{(n-1)!} + \frac{1}{n!}\right)x^n$$

$$= 1 + 8x + \frac{27}{2}x^2 + \sum_{n=3}^{\infty}\frac{(n+1)^3}{n!}x^n$$

$$= 1 + \frac{2^3}{1!}x + \frac{3^3}{2!}x^2 + \dots$$

Setting $x=1$,

$$15e = 1 + \frac{2^3}{1!} + \frac{3^3}{2!} + \frac{4^3}{3!} + \dots$$

STEP III 1997 Q2

$f(t) = \frac{\ln t}{t}$. For $t \to 0$, $\frac{\ln t}{t} \longrightarrow -\infty$.

For $t \to \infty$, $\frac{\ln t}{t} \longrightarrow 0$ as t grows quicker than $\ln t$.

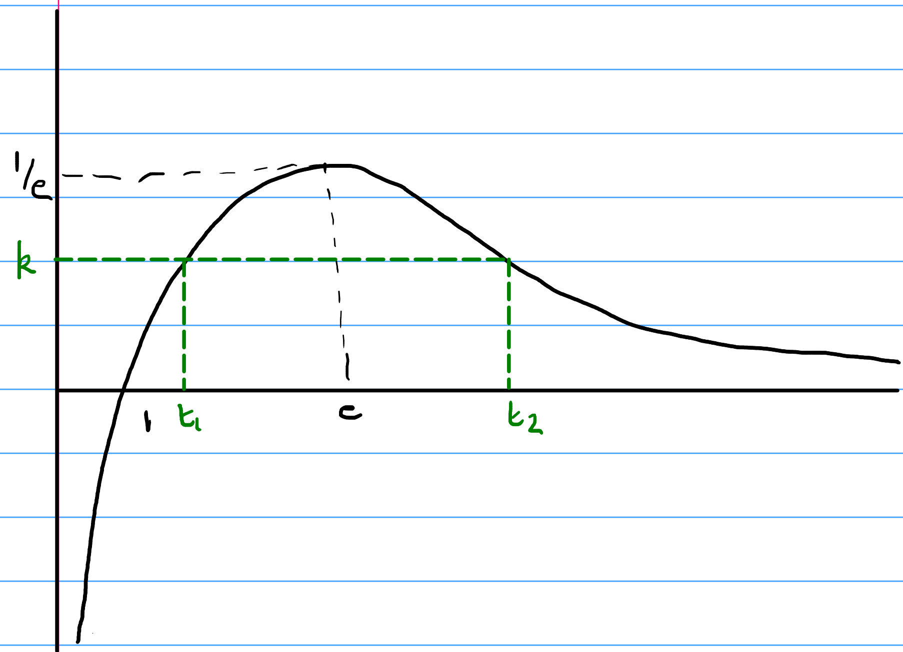
$\frac{\ln t}{t} = 0 \Rightarrow \ln t = 0 \Rightarrow t = 1$

$$\frac{df}{dt} = \frac{t \cdot \frac{1}{t} - \ln t}{t^2}$$

$$= \frac{1 - \ln t}{t^2}$$

stationary point is where $\frac{1-\ln t}{t^2} = 0 \Rightarrow t = e$

$f(e) = \frac{1}{e}$



If $f(t) = k$ is positive, there are exactly two values of t s.t. $f(t_1) = f(t_2) = k$, as shown on the diagram. When $f(t) = 1/e$ there is only one corresponding value of t (e).

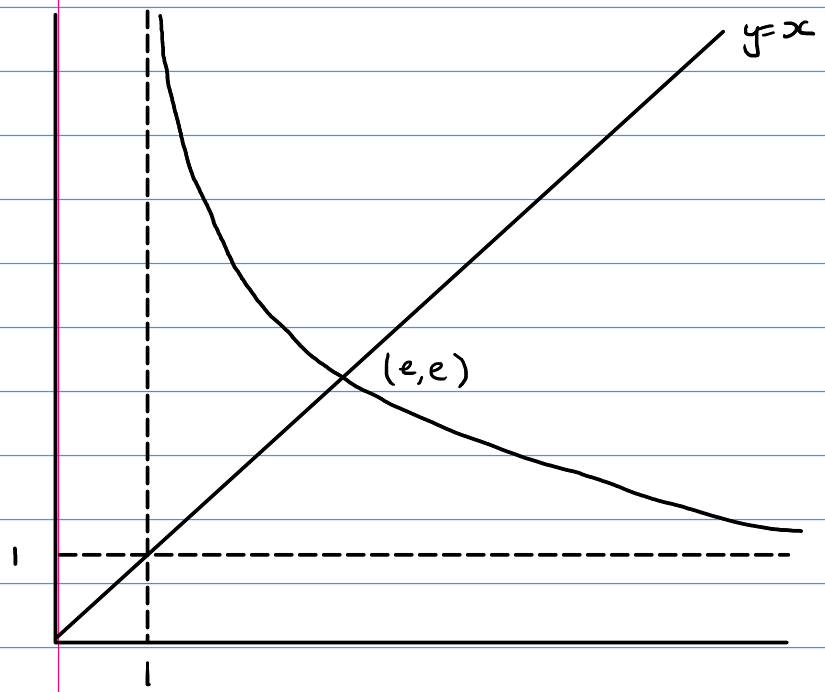
Now $x^y = y^x$

$\Leftrightarrow y \ln x = x \ln y$

$\Leftrightarrow \frac{\ln x}{x} = \frac{\ln y}{y}$

From before, we know there are two values of y satisfying $\frac{\ln y}{y} = \frac{\ln x}{x}$ for fixed $x>1$. One is x itself, and then another value. So two positive values of y satisfy $x^y = y^x$ for a fixed positive x (or one, if $x=e$).

To sketch this, $y=x$ works. Otherwise, consider the previous graph. For small positive $x>1$, $y>>0$. For $x=e$, $y=e$, and for large x, $y \approx 1$. So,



STEP III 1997 Q3

The roots of $z^n - 1 = 0$ are $1, \omega, \omega^2, \dots \omega^{n-1}$ (roots of unity).

Hence $(z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) = z^n - 1$

so, $(z-\omega)(z-\omega^2)\text{---}(z-\omega^{n-1}) = \dfrac{z^n-1}{z-1}$

But $1 + z + z^2 + \text{---} + z^{n-1} = \dfrac{z^n-1}{z-1}$ (sum of geometric series), so

$(z-\omega)(z-\omega^2)\text{---}(z-\omega^{n-1}) = 1 + z + z^2 + \text{---} + z^{n-1}$, as required.

Now $\overrightarrow{OA_i} = \omega^i \times e^{i\phi}$ for some fixed ϕ (as the A_i are rotated roots of unity).

so $\sum \overrightarrow{OA_i} = e^{i\phi} \sum \omega^i$

$= e^{i\phi}\left((\omega-\omega)(\omega-\omega^2)\text{---}(\omega-\omega^{n-1})\right)$ (by the above).

$= 0$, as required.

$|A_1A_k|$ is independent of ϕ, so wlog take $A_1 = (r, 0)$.

$$\sum_{k=1}^{n} |A_1A_k|^2 = \sum_{k=1}^{n} \left|r - re^{\frac{i(k-1)\pi}{n}}\right|^2$$

$$= r^2 \sum_{k=1}^{n} \left(1 - \cos\frac{(k-1)\pi}{n}\right)^2 + \sin^2\frac{(k-1)\pi}{n}$$

$$= r^2 \sum_{k=1}^{n} 2 - 2\cos\frac{(k-1)\pi}{n}$$

The final term vanishes, as it is $\mathrm{Re}\sum_{i=1}^{n} \omega^i = 0$, so

$$\sum_{k=1}^{n} |A_1A_k|^2 = r^2 \sum_{k=1}^{n} 2$$

$= 2r^2n$, as required.

STEP III 1997 Q4

$f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$ has distinct positive roots p, q, r, s.

(i) Suppose pqr, qrs, rsp, spq are not distinct. wlog, $pqr = qrs$. $q, r \neq 0$ (as roots are positive), so $p = s$ #.

(ii) $c = \sqrt[3]{\dfrac{pqr + qrs + rsp + spq}{4}} \geqslant \sqrt[3]{(p^3q^3r^3s^3)^{1/4}}$ (AM-GM inequality).

$= (pqrs)^{1/4}$

$= d$.

So $c > d$

(iii) f has four distinct roots, so there must exist a turning point between each adjacent pair of roots. There are 3 such pairs, so there are 3 turning points. But $f'(x)$ is a cubic, so has 3 roots - these must be the 3 (distinct) turning points.

(iv) $f'(x) = 4x^3 - 12ax^2 + 12bx - 4c^3$

$= 4(x^3 - 3ax^2 + 3bx - c^3)$

Suppose this has roots α, β, γ (which are distinct and positive, as they lie between the roots of $f(x)$).

Then $b = \sqrt{\dfrac{\alpha\beta + \beta\gamma + \gamma\alpha}{3}}$

$> \sqrt{(\alpha^2\beta^2\gamma^2)^{1/3}}$ (AM-GM inequality)

$= (\alpha\beta\gamma)^{1/3}$

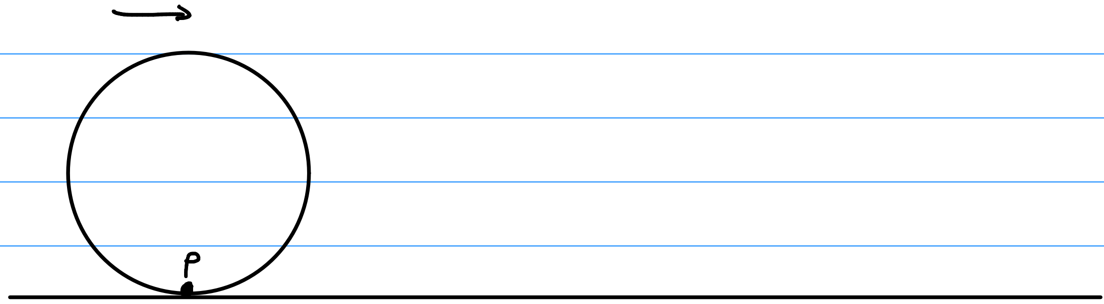
$= c$, so $b > c$.

(v) Similarly $f''(x)$ has two distinct positive roots, δ and ε.

$f''(x) = 12(x^2 - ax + b^2)$, so $a = \delta + \varepsilon > \sqrt{\delta\varepsilon}$

$= b$, as required.

STEP III 1997 Q5



Consider the point P. Its position as the circle rolls is

$x(t) = t - \sin t$

$y(t) = -\cos t$, where t ranging from 0 to 2π is the angle the wheel has rotated.

$$\text{Arc length} = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2(1 - 1 + 2\sin^2 t/2)}\, dt$$

$$= 2\int_0^{2\pi} |\sin t/2|\, dt$$

$$= 2\int_0^{2\pi} \sin t/2\, dt$$

$$= \left[-4\cos t/2\right]_0^{2\pi}$$

$$= 8.$$

So the ratio is $\frac{2\pi}{8} = \frac{\pi}{4}$.

STEP III 1997 Q6

$$(1-x^2)\frac{d^2y_n}{dx^2} - x\frac{dy_n}{dx} + n^2y_n = 0, \quad y_n(1)=1, \quad y_n(x) = (-1)^n y_n(-x)$$

Set $x = \cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}$$
$$= \frac{-1}{\sin\theta}\frac{dy}{d\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2}\frac{d\theta}{dx} + \frac{1}{\sin\theta\tan\theta}\frac{dy}{d\theta}\frac{d\theta}{dx}$$
$$= \frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2} - \frac{1}{\sin^2\theta\tan\theta}\frac{dy}{d\theta}$$

So the original equation becomes

$$\sin^2\theta \cdot \frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2} - \sin^2\theta\frac{1}{\sin^2\theta\tan\theta}\frac{dy}{d\theta} + \cos\theta \cdot \frac{1}{\sin\theta}\frac{dy}{d\theta} + n^2y_n = 0$$

$$\Rightarrow \frac{d^2y_n}{d\theta^2} + n^2y_n = 0$$

So $y_n = A_n\cos n\theta + B_n\sin n\theta$

Now $x = 1 \Rightarrow \cos\theta = 1$

$\Rightarrow \cos n\theta = 1$ and $\sin n\theta = 0$

So $y(1) = 1 \Rightarrow A_n = 1.$

So $y_n = \cos n\theta + B_n\sin n\theta$

Now $\cos(\theta) = \cos(-\theta) = x$, so $y_n(\theta) = y_n(-\theta) \Rightarrow B_n = 0.$

So $y_n(\theta) = \cos n\theta$

Then $y_0 = \cos(0\times\theta) = 1$, $y_1 = \cos\theta = x.$

$$y_{n+1} - 2xy_n + y_{n-1} = \cos(n+1)\theta - 2\cos\theta\cos n\theta + \cos(n-1)\theta$$
$$= \cos n\theta\cos\theta - \sin n\theta\sin\theta - 2\cos\theta\cos n\theta + \cos n\theta\cos\theta + \sin n\theta\sin\theta$$
$$= 0, \text{ as required.}$$

STEP III 1997 Q7

(i) $b_n = \frac{a}{1-r} = \frac{1/n+1}{1-1/n+1}$

$= \frac{1}{n+1-1}$

$= \frac{1}{n}$

(ii) Each term of a_n is positive, but smaller than the matching term in b_n. So $0 < a_n < b_n$.

(iii) $n!e = n!\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \text{—} + \frac{1}{n!} + \frac{1}{(n+1)!} + \text{—}\right)$

$= n!\left(\frac{1}{0!} + \frac{1}{1!} + \text{—} + \frac{1}{n!}\right) + \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \text{—}\right)$

$=$ an integer $+ a_n$

as $0 < a_n < b_n = 1/n < 1$, we have $n!e = [n!e] + a_n$

so $a_n = n!e - [n!e]$, as required.

(iv) Suppose e is rational, with $e = p/q$. Then taking $n = q$, $n!e = q! \cdot \frac{p}{q} = p(q-1)!$ is an integer. But then $n!e = [n!e] \Rightarrow a_n = 0$, but we have $a_n > 0$. So e is irrational.

STEP III 1997 Q8

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

(i) $$R_{-\alpha}AR_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} a\cos\alpha + b\sin\alpha & -a\sin\alpha + b\cos\alpha \\ b\cos\alpha + c\sin\alpha & -b\sin\alpha + c\cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} a\cos^2\alpha + 2b\sin\alpha\cos\alpha + c\sin^2\alpha & (c-a)\sin\alpha\cos\alpha + b(\cos^2\alpha - \sin^2\alpha) \\ (c-a)\sin\alpha\cos\alpha + b(\cos^2\alpha - \sin^2\alpha) & a\sin^2\alpha - 2b\sin\alpha\cos\alpha + c\cos^2\alpha \end{pmatrix}$$

To be diagonal, $(c-a)\sin\alpha\cos\alpha + b(\cos^2\alpha - \sin^2\alpha) = 0$

$$\Rightarrow \frac{(c-a)}{2}\sin 2\alpha + b\cos 2\alpha = 0$$

$$\Rightarrow \tan 2\alpha = \frac{2b}{a-c}$$

$$\Rightarrow \alpha = \frac{1}{2}\arctan\frac{2b}{a-c}$$

(ii) $$x^2 + (y + 2x\cot 2\theta)^2 = 1$$

$$\Rightarrow x^2 + y^2 + 4xy\cot 2\theta + 4x^2\cot^2 2\theta = 1$$

$$\Rightarrow x^2(1 + 4\cot^2 2\theta) + 4xy\cot 2\theta + y^2 = 1$$

$$(x \;\; y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2$$

$$\Rightarrow a = 1 + 4\cot^2 2\theta,\; b = 2\cot 2\theta,\; c = 1$$

So, $$\alpha = \frac{1}{2}\arctan\frac{4\cot 2\theta}{4\cot^2 2\theta}$$

$$= \frac{1}{2}\arctan(\tan 2\theta)$$

$$= \theta$$, as required.

The upper left element of the matrix is

$$a\cos^2\alpha + 2b\sin\alpha\cos\alpha + c\sin^2\alpha$$

$$= (1+4\cot^2 2\theta)\cos^2\theta + 4\cot 2\theta\sin\theta\cos\theta + \sin^2\theta$$

$$= \sin^2\theta + \cos^2\theta + 2\cot 2\theta \sin 2\theta + 4\cot^2 2\theta\cos^2\theta$$

$$= 1 + 2\cos 2\theta + \frac{\cos^2 2\theta}{\sin^2\theta\cos^2\theta}\cos^2\theta$$

$$= 1 + 2\cos 2\theta + \frac{\cos^2 2\theta}{\sin^2\theta}$$

$$= 1 + 2(1-2\sin^2\theta) + \left(\frac{1-2\sin^2\theta}{\sin\theta}\right)^2$$

$$= 1 + 2 - 4\sin^2\theta + (\operatorname{cosec}\theta - 2\sin\theta)^2$$

$$= 3 - 4\sin^2\theta + \operatorname{cosec}^2\theta - 4\sin\theta\operatorname{cosec}\theta + 4\sin^2\theta$$

$$= 3 + \operatorname{cosec}^2\theta - 4$$

$$= \operatorname{cosec}^2\theta - 1$$

$$= \underline{\cot^2\theta}$$

The bottom right element of the matrix is similarly

$$a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta$$

$$= (1+4\cot^2 2\theta)\sin^2\theta - 4\cot 2\theta\sin\theta\cos\theta + \cos^2\theta$$

$$= \sin^2\theta + \frac{4\cos^2 2\theta}{\sin^2 2\theta}\sin^2\theta - 2\frac{\cos 2\theta}{\sin 2\theta}\sin 2\theta + \cos^2\theta$$

$$= 1 + \frac{2(2\cos^2\theta-1)^2}{2\cos^2\theta} - 2(2\cos^2\theta - 1)$$

$$= 1 + \frac{1}{\cos^2\theta}(4\cos^4\theta - 4\cos^2\theta + 1) - 4\cos^2\theta + 2$$

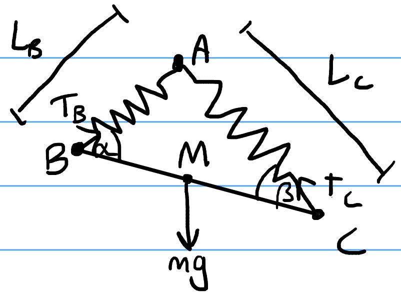
$$= 3 + 4\cos^2\theta - 4 + \sec^2\theta - 4\cos^2\theta$$

$$= \sec^2\theta - 1$$

$$= \tan^2\theta$$

(iii) The matrix $(x \;\; y)R_{-\alpha}AR_{\alpha}\binom{x}{y}$ is the ellipse with a new coordinate basis x' and y', with equation $(x'\cot\theta)^2 + (y'\tan\theta)^2 = 1$. Hence the axes have length $\cot\theta$ and $\tan\theta$.

STEP III 1997 Q9



Suppose the springs have natural lengths l_B and l_C, and stretched lengths L_B and L_C.

Taking moments about M, we have $T_B \sin\alpha = T_C \sin\beta$

$$\Rightarrow \frac{T_B}{T_C} = \frac{\sin\beta}{\sin\alpha}$$

But by the sine rule, $\frac{\sin\beta}{\sin\alpha} = \frac{L_B}{L_C}$, so $\frac{T_B}{T_C} = \frac{L_B}{L_C}$

$$\Rightarrow \frac{T_B}{L_B} = \frac{T_C}{L_C}$$

Now $T_B = K(L_B - l_B)$

so $l_B = L_B - \frac{1}{K}T_B$

Hence $\frac{l_B}{l_C} = \frac{L_B - \frac{1}{K}T_B}{L_C - \frac{1}{K}T_C}$

$$= \frac{KL_B - T_B}{KL_C - T_C}$$

$$= \frac{KL_B - \frac{T_B}{L_B}L_B}{KL_C - \frac{T_C}{L_C}L_C}$$

$$= \frac{KL_B - \frac{T_B}{L_B}L_B}{KL_C - \frac{T_B}{L_B}L_C}$$

$$= \frac{(K - T_B/L_B)L_B}{(K - T_B/L_B)L_C}$$

$$= \frac{L_B}{L_C}$$, as required.

STEP III 1997 Q10

Considering the horizontal movement, $-F = m\frac{dv}{dt}$

Considering the rotational movement, $-Fa = \frac{2}{5}ma^2\frac{dw}{dt}$

So, $\frac{2}{5}ma^2\frac{dw}{dt} - ma\frac{dv}{dt} = 0$

$\Rightarrow \frac{2}{5}a\frac{dw}{dt} - \frac{dv}{dt} = 0$

$\Rightarrow \frac{d}{dt}\left(\frac{2}{5}a\omega - v\right) = 0$

So $\frac{2}{5}a\omega - v$ is constant.

(i) To come to a complete stop, $\omega = v = 0 \Rightarrow \frac{2}{5}a\omega_0 - V_0 = 0$

$\Rightarrow \frac{V}{a\omega_0} = \frac{2}{5}$

(ii) Now $v = \frac{-V_0}{7}$, $a\omega = \frac{-V_0}{7}$

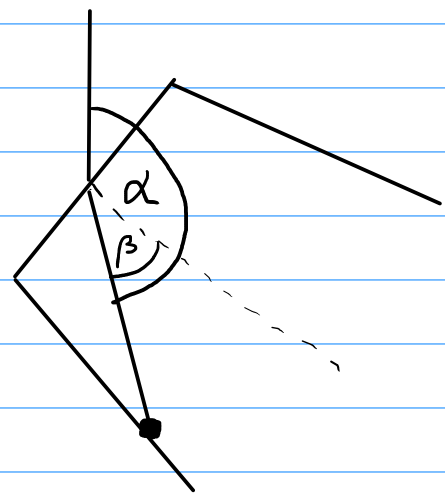
So $\frac{2}{5}a\omega - v = -\frac{2V_0}{35} + \frac{V_0}{7}$

$= \frac{3}{35}V_0$

So $\frac{2}{5}a\omega_0 - V_0 = \frac{3}{35}V_0$

$\Rightarrow \frac{V_0}{a\omega_0} = \frac{7}{19}$

STEP III 1997 Q11



At the point the clapper and bell seperate, they exert no force on each other, and have the same acceleration.

For the clapper: $mgl\sin\alpha = ml^2\ddot{\theta}$
For the bell : $Mgh\sin(\alpha-\beta) = Mk^2\ddot{\theta}$

$$\Rightarrow \frac{\sin\alpha}{l} = \frac{h\sin(\alpha-\beta)}{k^2}$$

$$\Rightarrow \sin\alpha = \frac{hl}{k^2}\sin(\alpha-\beta)$$
$$= \frac{hl}{k^2}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$\div \sin\alpha\sin\beta$

$$\operatorname{cosec}\beta = \frac{hl}{k^2}(\cot\beta - \cot\alpha)$$

$$\Rightarrow \cot\alpha = \cot\beta - \frac{k^2}{hl}\operatorname{cosec}\beta$$

STEP III 1997 Q12

(i) $P(K=k) = q^{k-1}p$

so $G(s) = \sum_{k=1}^{\infty} q^{k-1} p s^k$

$= \frac{p}{q} \sum_{k=1}^{\infty} (qs)^k$

$= \frac{p}{q} \cdot \frac{qs}{1-qs}$

$= \frac{ps}{1-qs}.$

$G'(s) = \frac{p(1-qs)+qps}{(1-qs)^2}$

$= \frac{p}{(1-qs)^2}$

$EK = G'(1) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = 1/p.$

$G''(s) = 2pq(1-qs)^{-3}$

$G''(1) = \frac{2pq}{p^3} = \frac{2q}{p^2}$

$G''(1) = E(K^2-K)$

so $EK^2 = G''(1) + EK$

$= \frac{2q}{p^2} + \frac{1}{p}$

$\text{Var}K = EK^2 - (EK)^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2}$

$= \frac{2q+p-1}{p^2}$

$= \frac{2-2p+p-1}{p^2}$

$= \frac{1-p}{p^2}$

(ii) When there are i cards left to find, there is an $\frac{i}{52}$ probability of success, and a $\frac{52-i}{52}$ chance of failure. Setting N_i as the time to find the next card where there are i cards left, we have

$$EN = \sum_{i=1}^{52} EN_i$$
$$= \sum_{i=1}^{52} \frac{52}{i}$$
$$\approx 52(0.5 + \ln 52)$$

Now $e^2 \approx 2.7^2 = 7.29 \approx 7.3$
$e^3 \approx 2.7 \times 7.3 = 19.71 \approx 19.7$
$e^4 \approx 2.7 \times 19.7 = 53.19 \approx 52$

so $\ln 52 \approx 4$.

So $EN \approx 52(0.5 + 4)$
$= 234$

STEP III 1997 Q13

(i) $\mathrm{E}e^{\theta X} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} e^{\theta x}\, dx$

Now $-\frac{1}{2}x^2 + \theta x = -\frac{1}{2}\left[x^2 - 2\theta x\right]$

$= -\frac{1}{2}\left[(x-\theta)^2 - \theta^2\right]$

$= -\frac{1}{2}(x-\theta)^2 + \frac{1}{2}\theta^2$

so $\mathrm{E}e^{\theta X} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} e^{\frac{1}{2}\theta^2}\, dx$

$= e^{\frac{1}{2}\theta^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}\, dx$

$= e^{\frac{1}{2}\theta^2}$ as integral is 1 as pdf of normal distribution.

(ii) $\mathrm{E}e^{\theta(aX+bY)}$

$= \mathrm{E}e^{a\theta X}\, \mathrm{E}e^{b\theta Y}$

$= e^{\frac{1}{2}(a\theta)^2} e^{\frac{1}{2}(b\theta)^2}$

$= e^{\frac{1}{2}\theta^2(a^2+b^2)}$

so we require $a^2+b^2=1$.

(iii) $\mathrm{E}Z^\theta = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} e^{\theta(\mu+\sigma x)}\, dx$

$= e^{\mu\theta} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} e^{\theta\sigma x}\, dx$

$= e^{\mu\theta} e^{\frac{1}{2}(\theta\sigma)^2}$

$= e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}$

(iii) $\mathrm{E}Z^1 = e^{\mu + \frac{1}{2}\sigma^2}$

$\mathrm{E}Z^2 = e^{2\mu + 2\sigma^2}$

So $\mathrm{Var}\,Z = e^{2\mu + 2\sigma^2} - \left(e^{\mu + \frac{1}{2}\sigma^2}\right)^2$

$= e^{2\mu}\left(e^{2\sigma^2} - e^{\sigma^2}\right)$

$= e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$

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Perimeter $= 2L + 2W$

$EP = 2\mu_1 + 2\mu_2$

$\text{Var}P = 4\sigma_1^2 + 4\sigma_2^2$

Area $= L \times W$

$EA = \mu_1\mu_2$ by independence

$EA^2 = EL^2 EW^2$

$= (\mu_1^2 + \sigma_1^2)(\mu_2^2 + \sigma_2^2)$

So $\text{Var}A = (\mu_1^2 + \sigma_1^2)(\mu_2^2 + \sigma_2^2) - \mu_1^2\mu_2^2$

Now $EPEA = 2(\mu_1 + \mu_2)\mu_1\mu_2$

$EPA = E[(2L+2W)(LW)]$

$= E(2L^2W + 2LW^2)$

$= 2EL^2EW + 2ELEW^2$ by independence

$= 2(\mu_1^2 + \sigma_1^2)\mu_2 + 2(\mu_2^2 + \sigma_2^2)\mu_1$

$= 2(\mu_1 + \mu_2)\mu_1\mu_2 + 2\mu_2\sigma_1^2 + 2\mu_1\sigma_2^2$

$= EPEA + 2\mu_2\sigma_1^2 + 2\mu_1\sigma_2^2$

$\neq EPEA$, so P and A are not independent.