STEP II 1997 Q1

The possibilites for each digit, and their frequency, are as follows

$$\begin{array}{ll} 2000 & (\times 64) \\ \\ 5000 & (\times 64) \end{array} \qquad \left.\begin{array}{c} 0 \\ 200 \\ 500 \\ 700 \end{array}\right\} 32 \text{ each} \qquad \left.\begin{array}{c} 0 \\ 20 \\ 50 \\ 70 \end{array}\right\} 32 \text{ each} \qquad \left.\begin{array}{c} 0 \\ 2 \\ 5 \\ 7 \end{array}\right\} 32 \text{ each}$$

So the sum is

$$64(2000+5000)+32(0+200+500+700)+32(0+20+50+70)+32(0+2+5+7)$$
$$=32(14{,}000+1{,}400+140+14)$$
$$=2^5\times 14(1000+100+100+1)$$
$$=2^6\times 7\times 1111$$
$$=2^6\times 7\times 11\times 101$$

STEP II 1997 Q2

$3 = \frac{2}{x_1} \Rightarrow x_1 = 2/3$

$\frac{2}{3} + \frac{2}{x_2} = 3 \Rightarrow \frac{2}{x_2} = \frac{7}{3}$

$\Rightarrow x_2 = 6/7$

$\frac{6}{7} + \frac{2}{x_3} = 3 \Rightarrow \frac{2}{x_3} = \frac{15}{7}$

$\Rightarrow x_3 = \frac{14}{15}$

Hypothesis: $x_n = \frac{2^{n+1}-2}{2^{n+1}-1}$

Proof by induction.

For $n=1$, $x_1 = \frac{2^2-2}{2^2-1}$

$= 2/3$ ✓

Assume true for $n=k$. Then

$$3 = \frac{2^{k+1}-2}{2^{k+1}-1} + \frac{2}{x_{k+1}}$$

$$\Rightarrow \frac{2}{x_{k+1}} = \frac{3(2^{k+1}-1)-(2^{k+1}-2)}{2^{k+1}-1} = \frac{2\cdot 2^{k+1}-1}{2^{k+1}-1}$$

$$= \frac{2^{k+2}-1}{2^{k+1}-1}$$

$$\Rightarrow x_{k+1} = \frac{2(2^{k+1}-1)}{2^{k+2}-1}$$

$$= \frac{2^{k+2}-2}{2^{k+2}-1}$$

True for $n=1$, and if true for $n=k$ then true for $n=k+1$, so true for all $n \in \mathbb{N}$.

STEP II 1997 Q3

Note that $(x^2+2+2x)(x^2+2-2x)$

$= (x^2+2)^2 - (2x)^2$

$= x^4+4$

So $(ax+b)(x^2-2x+2) + (cx+d)(x^2+2x+2) \equiv 1$

$\Rightarrow x^3(a+c) + x^2(b-2a+d+2c) + x(2a-2b+2c+2d) + (2b+2d) = 1$

We have $a=-c$

$2b = 1-2d$

$2a-2b+2c+2d = 0 \Rightarrow -2b+2d = 0$

$\Rightarrow 4d-1=0$

$\Rightarrow d = 1/4$

$\Rightarrow b = 1/4$

$b-2a+d+2c = 0 \Rightarrow 4c + 1/2 = 0$

$\Rightarrow c = -1/8$

$\Rightarrow a = 1/8$

So $\dfrac{1}{x^4+4} = \dfrac{\frac{1}{8}x+\frac{1}{4}}{x^2+2x+2} + \dfrac{-\frac{1}{8}x+\frac{1}{4}}{x^2-2x+2}$

Hence $\displaystyle\int_0^1 \frac{1}{x^4+4}\,dx = \int_0^1 \frac{\frac{1}{8}x+\frac{1}{4}}{x^2+2x+2} + \frac{-\frac{1}{8}x+\frac{1}{4}}{x^2-2x+2}\,dx$

$\displaystyle = \int_0^1 \frac{1}{16}\frac{2x+2}{x^2+2x+2} + \frac{1}{16}\frac{2}{x^2+2x+2} - \frac{1}{16}\frac{2x-2}{x^2-2x+2} + \frac{1}{16}\frac{2}{x^2-2x+2}\,dx$

$\displaystyle = \frac{1}{16}\int_0^1 \frac{2x+2}{x^2+2x+2}\,dx + \frac{1}{8}\int_0^1 \frac{1}{(x+1)^2+1}\,dx - \frac{1}{16}\int_0^1 \frac{2x-2}{x^2-2x+2}\,dx + \frac{1}{8}\int_0^1 \frac{1}{(x-1)^2+1}\,dx$

$$= \left[\frac{1}{16}\ln|x^2+2x+2\right]_0^1 + \left[\frac{1}{8}\arctan(x+1)\right]_0^1 - \left[\frac{1}{16}\ln|x^2-2x+2|\right]_0^1 + \left[\frac{1}{8}\arctan(x-1)\right]_0^1$$

$$= \frac{1}{16}\ln 5 + \frac{1}{16}\ln 2 + \frac{1}{8}\arctan 2 - \frac{1}{8}\arctan 1 - \frac{1}{16}\ln 2 + \frac{1}{8}\arctan 1$$

$= \frac{1}{16}\ln 5 + \frac{1}{8}\arctan 2$, as required.

STEP II 1997 Q4

We have $p(x) = (x-a)q(x) + r$

$\Rightarrow \quad p(a) = (a-a)q(a) + r$

$\Rightarrow \quad p(a) = r$

(i) $p(1) = 3 \Rightarrow r(1) = 3$

$p(2) = 1 \Rightarrow r(2) = 1$

$p(3) = 5 \Rightarrow r(3) = 5$

$r(x) = ax^2 + bx + c$. Using the above,

$a + b + c = 3 \quad (1)$

$4a + 2b + c = 1 \quad (2)$

$9a + 3b + c = 5 \quad (3)$

$(2) - (1) \Rightarrow 3a + b = -2 \quad (4)$

$(3) - (1) \Rightarrow 8a + 2b = 2 \quad (5)$

$(5) - 2 \times (4) \Rightarrow 2a = 6$

$\Rightarrow a = 3$

$b = -2 - 3a = -11$

$c = 3 - a - b = 11$

So $r(x) = 3x^2 - 11x + 11$

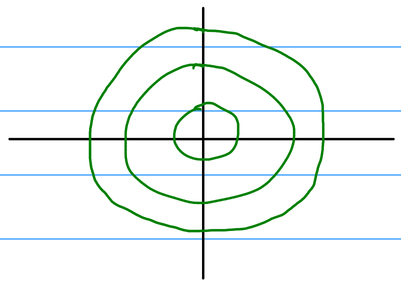
(ii) $P(x) = x + x(x-1)(x-2) \_\_\_\_ (x-n)$

STEP II 1997 Q5

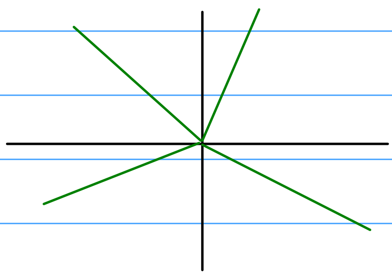
$z = e^u(\cos v + i\sin v)$

$z = ie \Rightarrow u = 1$, $\cos v = 0$, $\sin v = 1$, so $w = 1 + (2n + \frac{1}{2}\pi)i$ for $n \in \mathbb{Z}$

$u = \text{constant} \Rightarrow z = Ae^{iz}$ for constant A, or circles centred at the origin of radius u.



$v = \text{constant} \Rightarrow z = e^u \cdot e^{i\theta}$ for fixed θ – half lines from the origin.



Now z is in the first quadrant. So $\cos v > 0$, $\sin v > 0$ (and $u \in \mathbb{R}$). Could have $0 < v < \pi/2$, or $2\pi < v < 5\pi/2$.

So $\{u \in \mathbb{R}, 0 < v < \pi/2\}$ or $\{u \in \mathbb{R}, 2\pi < v < 5\pi/2\}$

Now $0 < |z| < 1$, so $e^u < 1$, so $u < 0$. We also need to restrict v to a 2π length interval to ensure a one-to-one correspondence.

So $\{u < 0, 0 < v < 2\pi\}$ or $\{u < 0, 2\pi < v < 4\pi\}$

STEP II 1997 Q6

$$\tan^2\phi = 2\tan\phi + 1 \Rightarrow 2\tan\phi = \tan^2\phi - 1$$
$$\Rightarrow \frac{2\tan\phi}{1-\tan^2\phi} = -1$$
$$\Rightarrow \tan 2\phi = -1$$

Now $\tan\theta = 2 + \tan 3\theta$

$$\Rightarrow \tan\theta = 2 + \frac{\tan 2\theta + \tan\theta}{1 - \tan\theta\tan 2\theta}$$

then writing $\tan\theta = t$

$$= 2 + \frac{\frac{2t}{1-t^2} + t}{1 - \frac{2t^2}{1-t^2}}$$

$$t = 2 + \frac{3t - t^3}{1-3t^2}$$
$$\Rightarrow t - 3t^3 = 2 - 6t^2 + 3t - t^3$$
$$\Rightarrow 0 = 2t^3 - 6t^2 + 2t + 2$$
$$\Rightarrow t^3 - 3t^2 + t + 1 = 0$$

$t = 1$ is a solution.

$$\begin{array}{r|l}
 & t^2 - 2t - 1 \\
\hline
t-1 & t^3 - 3t^2 + t + 1 \\
 & \underline{t^3 - t^2} \\
 & \quad -2t^2 + t \\
 & \quad \underline{-2t^2 + 2t} \\
 & \qquad\quad -t + 1
\end{array}$$

so we have $(t-1)(t^2-2t-1)$

So $\tan\theta = 1$ or $\tan^2\theta = 2\tan\theta + 1 \Rightarrow \tan 2\theta = -1$

So $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ $\qquad\qquad \theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

So $\theta = \pi/4,\ 3\pi/8,\ 7\pi/8,\ 5\pi/4,\ 11\pi/8,\ 15\pi/8$

Now $\cot\theta = 2 + \cot 3\theta$

$$\Rightarrow \frac{1}{t} = 2 + \frac{1-3t^2}{3t-t^3}$$

$$\Rightarrow 1 = 2t + \frac{1-3t^2}{3-t^2}$$

$$\Rightarrow 3 - t^2 = 6t - 2t^3 + 1 - 3t^2$$

$$\Rightarrow 2t^3 + 2t^2 - 6t + 2 = 0$$

$$\Rightarrow t^3 + t^2 - 3t + 1 = 0$$

As before, $t=1$ works.

$$\begin{array}{r|l} & t^2 + 2t - 1 \\ \hline t-1 & t^3 + t^2 - 3t + 1 \\ & \underline{t^3 - t^2} \\ & \quad 2t^2 - 3t \\ & \quad \underline{2t^2 - 2t} \\ & \qquad -t + 1 \end{array}$$

So $(t-1)(t^2+2t-1) = 0$

So $t - 1 = 0$ or $\frac{2t}{1-t^2} = 1$

$$\Rightarrow \tan\theta = 1 \quad \text{or} \quad \tan 2\theta = 1$$

$$\Rightarrow \theta = \pi/4, -3\pi/4 \quad \text{or} \quad \theta = \pi/8, -3\pi/8, -7\pi/8, -11\pi/8$$

$$\Rightarrow \theta = -11\pi/8, -7\pi/8, -3\pi/4, -3\pi/8, \pi/8, \pi/4$$

STEP II 1997 Q7

$y^2 = x^2(a^2 - x^2)$

Max and min values of y occur at the maximum of y^2 (the equation is symmetric in $y \to -y$).

We make the substitution $u = y^2$, $v = x^2$, so we try to maximise u as a function of v.

$u = v(a^2 - v)$

$\Rightarrow \frac{du}{dv} = a^2 - 2v = 0$

$\Rightarrow v = a^2/2$

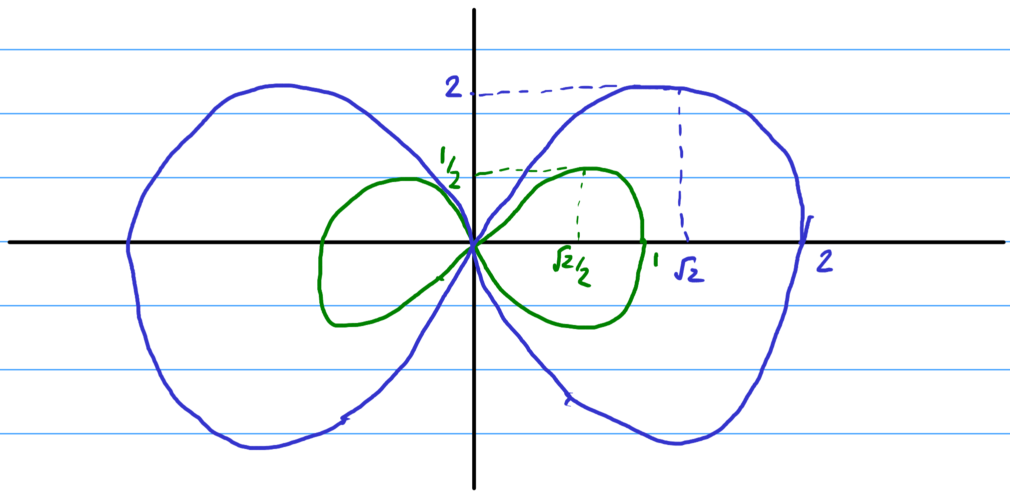
$\Rightarrow x^2 = a^2/2$

Here $y^2 = a^2/2(a^2 - a^2/2)$

$= a^4/4$

So $y_{max} = a^2/2$, $y_{min} = -a^2/2$

Now, the zeroes of the function are at $x = 0$, $x = \pm a$.



STEP II 1997 Q8

$\int_a^b f(t)\,dt$ is the limit of splitting the interval into rectangles and finding the area of the rectangles. If $f(t) \geq g(t)$ on that interval, then the rectangles are taller and have larger area, so $\int_a^b f(t) \geq \int_a^b g(t)\,dt$.

Consider $f(t) = t^{p-1}$, $g(t) = t^{q-1}$, whith $p > q > 0$. Then, for $t \geq 1$, clearly $f(t) \geq g(t)$.

Hence for $x \geq 1$, $\int_1^x t^{p-1}\,dt \geq \int_1^x t^{q-1}\,dt$

$$\Rightarrow \left[\frac{t^p}{p}\right]_1^x \geq \left[\frac{t^q}{q}\right]_1^x$$

$$\Rightarrow \frac{t^p - 1}{p} \geq \frac{t^q - 1}{q}.$$

With f, g, p, q as before, for $0 \leq x \leq 1$, we have

$$f(t) \leq g(t)$$

$$\Rightarrow \int_x^1 t^{p-1}\,dt \leq \int_x^1 t^{q-1}\,dt$$

$$\Rightarrow \frac{1-t^p}{p} \leq \frac{1-t^q}{q}$$

$$\Rightarrow \frac{t^p - 1}{p} \geq \frac{t^q - 1}{q}$$

Now take $f(t) = \frac{t^p - 1}{p}$, $g(t) = \frac{t^q - 1}{q}$ with $p > q > 0$.

Then, from above, $f(t) \geq g(t)$ for $x \geq 0$

$$\Rightarrow \int_0^x f(t)\,dt \geq \int_0^x g(t)\,dt$$

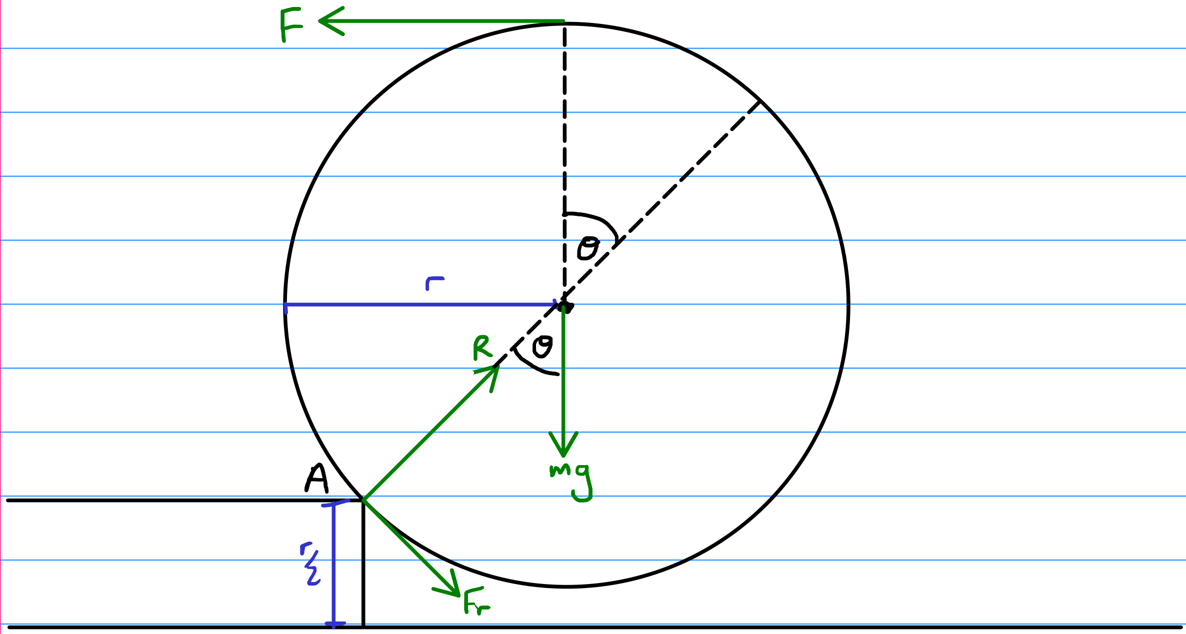
$$\Rightarrow \int_0^x \frac{t^p - 1}{p}\,dt \geq \int_0^x \frac{t^q - 1}{q}\,dt$$

$$\Rightarrow \frac{1}{p}\left[\frac{t^{p+1}}{p+1} - t\right]_0^x \geqslant \frac{1}{q}\left[\frac{t^{q+1}}{q+1} - t\right]_0^x$$

$$\Rightarrow \frac{1}{p}\left(\frac{x^{p+1}}{p+1} - x\right) \geqslant \frac{1}{q}\left(\frac{x^{q+1}}{q+1} - x\right)$$

$$\Rightarrow \frac{1}{p}\left(\frac{x^{p}}{p+1} - 1\right) \geqslant \frac{1}{q}\left(\frac{x^{q}}{q+1} - 1\right)$$

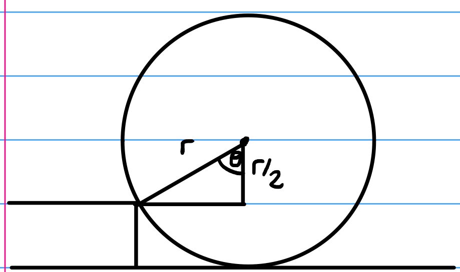
STEP II 1997 Q9



The motion is slow, so consider as a statics problem.

$$M(A): F(r + r\cos\theta) = mgr\sin\theta$$
$$\Rightarrow F(1+\cos\theta) = mg\sin\theta$$
$$\Rightarrow F = mg\frac{\sin\theta}{1+\cos\theta}$$

Now $\sin\theta$ is increasing in θ, $1+\cos\theta$ is decreasing in θ, so $F = mg\frac{\sin\theta}{1+\cos\theta}$ is increasing in θ. Thus its maximum value is at the maximum value of θ, which is at the start of the motion.



$\cos\theta = \frac{1}{2} \Rightarrow \sin\theta = \sqrt{3}/2$,

so $F = mg\dfrac{\sqrt{3}/2}{1+1/2}$

$= mg\sqrt{3}/3$

$= \dfrac{mg}{\sqrt{3}}$.

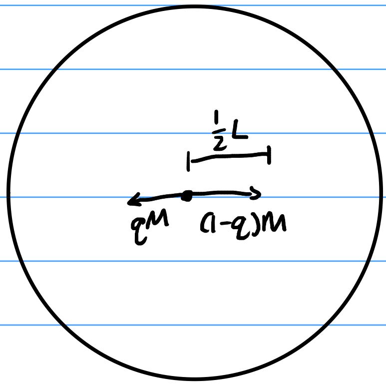
Taking moments about the centre of the circle, $F = F_r$. So the maximum friction is $mg/\sqrt{3}$.

Resolving parallel to R, we have

$$\begin{aligned} R &= mg\cos\theta + F\sin\theta \\ &= mg\cos\theta + mg\frac{\sin^2\theta}{1+\cos\theta} \\ &= mg\left(\cos\theta + \frac{1-\cos^2\theta}{1+\cos\theta}\right) \\ &= mg(\cos\theta + 1 - \cos\theta) \\ &= mg \end{aligned}$$

So $R = mg$. Since maximum friction is $mg/\sqrt{3}$, we hence must have $\mu \geqslant 1/\sqrt{3}$.

STEP II 1997 Q10



Conservation of momentum perpendicular to the axis of the cylinder:

$$qMv_1 = (1-q)Mv_2 \implies \frac{v_1}{v_2} = \frac{1-q}{q}$$

The first particle travels $\frac{5}{2}L$ in the time the second travels $\frac{1}{2}L$, so

$$\frac{v_1}{v_2} = \frac{5}{3}$$

$$\implies \frac{1-q}{q} = \frac{5}{3}$$

$$\implies 3 - 3q = 5q$$

$$\implies q = 3/8.$$

$$v_1 = \frac{\frac{5}{2}L}{\frac{5}{v}L} = \frac{v}{2}, \quad v_2 = \frac{3}{5}v_1 = \frac{3}{10}v$$

So KE gain $= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

$$= \frac{1}{2}Mv^2\left(\frac{3}{8}\left(\frac{1}{2}\right)^2 + \frac{5}{8}\left(\frac{3}{10}\right)^2\right)$$

$$= \frac{1}{2}Mv^2\left(\frac{3}{32} + \frac{9}{160}\right)$$

$$= \frac{3Mv^2}{40}$$

After coalescence, momentum perpendicular to axis is still zero, so velocity is just v parallel to the axis of the cylinder.

STEP II 1997 Q11

To net:

s	$H-h$
u	$V\sin\alpha$
v	$\times$
a	g
t	t_1

To ground:

s	H
u	$V\sin\alpha$
v	$\times$
a	g
t	t_2

$$H-h = V\sin\alpha t_1 + \frac{1}{2}gt_1^2 \quad (1)$$

$$H = V\sin\alpha t_2 + \frac{1}{2}gt_2^2 \quad (2)$$

Further, $t_1 = \frac{a}{V\cos\alpha}$ $\quad t_2 = \frac{a+b}{V\cos\alpha}$

So (1) becomes

$$H-h = a\tan\alpha + \frac{ga^2}{2V^2\cos^2\alpha}$$

(2) becomes

$$H = (a+b)\tan\alpha + \frac{g(a+b)^2}{2V^2\cos^2\alpha}$$

So (2) gives us

$$2V^2(H-(a+b)\tan\alpha) = g(a+b)^2\sec^2\alpha$$

$$\Rightarrow V^2 = \frac{g(a+b)^2(1+\tan^2\alpha)}{2[H-(a+b)\tan\alpha]}$$

Substituting this into (1),

$$H-h = a\tan\alpha + \frac{\cancel{g}a^2(1+\cancel{\tan^2\alpha})}{\cancel{2}} \cdot \frac{\cancel{2}[H-(a+b)\tan\alpha]}{\cancel{g}(a+b)^2(1+\cancel{\tan^2\alpha})}$$

$$= a\tan\alpha + \frac{a^2}{(a+b)^2}H - \frac{a^2}{(a+b)}\tan\alpha$$

$$\Rightarrow \tan\alpha\left(\frac{a^2}{a+b} - a\right) = \left(\frac{a^2}{(a+b)^2} - 1\right)H + h$$

$$\Rightarrow \tan\alpha\left(\frac{-ab}{a+b}\right) = \left(\frac{a^2-(a+b)^2}{(a+b)^2}\right)H + h$$

$$\Rightarrow \tan\alpha = \frac{2ab+b^2}{ab(a+b)}H - \frac{a+b}{ab}h$$

$$\Rightarrow \tan\alpha = \frac{2a+b}{a(a+b)}H - \frac{a+b}{ab}h$$

Now $v^2 > 0$, $\tan\alpha > 0$.

Hence $H > (a+b)\tan\alpha$

$$= (a+b)\left[\frac{2a+b}{a(a+b)}H - \frac{a+b}{ab}h\right]$$
$$= \frac{2a+b}{a}H - \frac{(a+b)^2}{ab}h$$

So $H\left(\frac{2a+b}{a} - 1\right) < \frac{(a+b)^2}{ab}h$

$$\Rightarrow H\left(\frac{a+b}{a}\right) < \frac{(a+b)^2}{ab}h$$

$$\Rightarrow H < \frac{a+b}{b}h$$

Also, $\frac{2a+b}{a(a+b)}H > \frac{a+b}{ab}h$

$$\Rightarrow H > \frac{(a+b)^2}{b(2a+b)}h$$

So $\frac{a+b}{b}h < H < \frac{(a+b)^2}{b(2a+b)}h$

STEP II 1997 Q12

We need either 0 or 2 fails, which occurs with probability

$$(2/3)^3 + 3 \times (2/3) \times (1/3)^2$$
$$= \frac{8}{27} + \frac{6}{27}$$
$$= 14/27$$

Note $(a+b)^n - (a-b)^n = 2(a^n + \binom{n}{2}a^{n-2}b^2 + \binom{n}{4}a^{n-4}b^4 + \dots)$

The probability of success is the probability of an even number of fails around the circle, so if $X \sim B(n, p)$ this is

$$P(X=0) + P(X=2) + P(X=4) + \dots$$
$$= p^n + \binom{n}{2}p^{n-2}(1-p)^2 + \binom{n}{4}p^{n-4}(1-p)^4 + \dots$$
$$= \tfrac{1}{2}(p+(1-p))^n + (p-(1-p))^n$$
$$= \tfrac{1}{2}\left[1 + (2p-1)^n\right]$$

Let Y be the number of games, and $p_n = \frac{1}{2}[1+(2p-1)^n]$

Then $P(Y=k) = (1-p_n)^{k-1}p_n$

So $EY = \sum_{k=1}^{\infty} p_n(1-p_n)^{k-1} \cdot k$

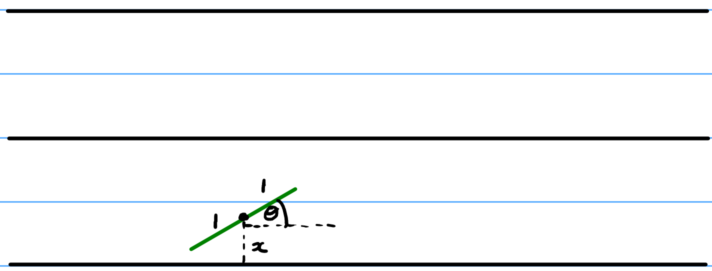
$$= p_n \sum_{k=1}^{\infty} k \cdot (1-p_n)^{k-1}$$

$$= p_n \cdot p_n^{-2}$$

$$= \frac{1}{p_n}$$

$$= \frac{2}{1+(2p-1)^n}$$

STEP II 1997 Q13



i) The needle crosses the line if $\frac{x}{\sin\theta} < 1 \iff \sin\theta > x$

So $P(\text{crosses line}) = P(\sin\theta > x)$

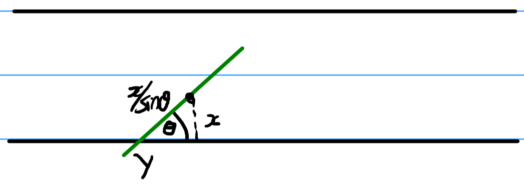
$$= P(\theta > \arcsin x)$$

But $\theta \sim U[0, \pi/2]$, so this is

$$\frac{\pi/2 - \arcsin x}{\pi/2}$$

$$= \frac{\pi - 2\arcsin x}{\pi}$$

ii)



$$P(Y<y \mid x<\sin\theta) = P\left(1 - \frac{x}{\sin\theta} < y \,\middle|\, x<\sin\theta\right)$$

$$= P\left(\sin\theta < \frac{x}{1-y} \,\middle|\, \sin\theta > x\right)$$

$$= P\left(x < \sin\theta < \frac{x}{1-y}\right) / P(x<\sin\theta)$$

$$= \frac{\arcsin\frac{x}{1-y} - \arcsin x}{\pi/2} \div \frac{\arcsin x}{\pi/2}$$

$$= \frac{\arcsin\frac{x}{1-y} - \arcsin x}{\arcsin x}$$

iii) $X \sim U[0,1]$, $\theta \sim U[0, \pi/2]$

$$P(\text{crosses line}) = \int_0^1 P(\text{crosses line} \mid X = x)\,dx \qquad (\text{as } X \sim U[0,1])$$

$$= \int_0^1 \frac{2\arcsin x}{\pi}\,dx$$

$$= \frac{2}{\pi}\int_0^1 \arcsin x\,dx$$

u $\arcsin x$ $\quad$ v' 1

u' $\frac{1}{\sqrt{1-x^2}}$ $\quad$ v x

$$= \frac{2}{\pi}\Big[x\arcsin x\Big]_0^1 - \frac{2}{\pi}\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{2}{\pi}\Big[x\arcsin x + \sqrt{1-x^2}\Big]_0^1$$

$$= \frac{2}{\pi}\left(\pi/2 - 1\right)$$

$$= 1 - 2/\pi$$

STEP II 1997 Q14

It takes me $\frac{9600}{30} = 320$ seconds to drive through the tunnel. It takes other cars $\frac{9600}{32} = 300$ seconds. So any cars which enter within 20 seconds of me will catch me up.

On average 6 cars enter per minute, so the mean number in 20 seconds is $\frac{6}{3} = 2$.

Now, for two vehicles to be queueing behind me when I leave, we either have 2 vehicles entering in 20 seconds, both travelling at $32ms^{-1}$, or more than 2 entering with the first two travelling at $32ms^{-1}$ and the third at $30ms^{-1}$.

The probability of this is

$$\frac{e^{-2}\times 2^2}{2!}\times\left(\frac{1}{2}\right)^2 + \left(1-e^{-2}\left(1+2+\frac{2^2}{2!}\right)\right)\cdot\left(\frac{1}{2}\right)^3$$
$$= \frac{e^{-2}}{2} + \frac{1}{8}\left(1-5e^{-2}\right)$$
$$= \frac{1-e^{-2}}{8}$$