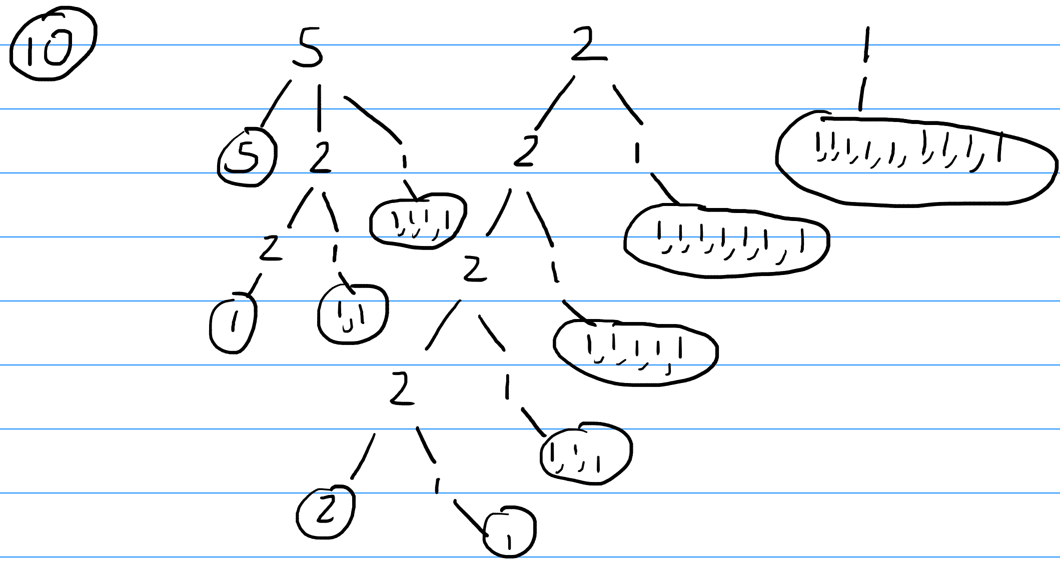
STEP I 1997 Q1

At each stage, consider the largest coin not yet accounted for.



So there are 11 ways.

For 20p, again consider the largest coin.

20	1	way
10	11	ways (as found previously, 10p left.
5	1	way with 5555
	3	ways with 555 (then 221, 2111, 11111)
	6	ways with 55 (the descendants of 2 and 1 above)
	8	ways with 5 (0,1,2,...,or 7 2s, and the rest 1)
2	10	ways (choose the number of 2s, the rest 1)
1	1	way
	41	ways in total.

STEP I 1997 Q2

(i) $f(x) = \arctan x + \arctan\left(\frac{1-x}{1+x}\right)$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \times \frac{(-1+x)-(1-x)}{(1+x)^2}$

$= \frac{1}{1+x^2} - \frac{2}{(1+x)^2+(1-x)^2}$

$= \frac{1}{1+x^2} - \frac{2}{2(1+x^2)}$

$= 0$

So f is constant. $f(0) = \arctan 0 + \arctan 1$

$= 0 + \pi/4$

$= \pi/4$

So $f(x) = \pi/4$ for all x.

(ii) $x = y\sin y^2$

$\frac{dx}{dy} = \sin y^2 + 2y^2\cos y^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin y^2 + 2y^2\cos y^2}$

$= \frac{1}{\left(\frac{x}{y}\right) + 2y^2\sqrt{1-\left(\frac{x}{y}\right)^2}}$

$= \frac{y}{x+2y^2\sqrt{y^2-x^2}}$, as required.

STEP I 1997 Q3

(i) $a_1 = 3$, $a_{n+1} = a_n^3$.

$a_2 = 27$, which has units digit 7. $7^3 = 343$, so the units digit of a_3 is 3.

(ii) We want to show $a_7 = (((((3^3)^3)^3)^3)^3)^3 > 10^{100}$

$$\Leftrightarrow 3\log((((3^3)^3)^3)^3)^3 > 100\log 10$$

$$\Leftrightarrow 3^6 \log 3 > 100 \log 10$$

$$\Leftrightarrow \frac{3^6}{100} > \frac{\log 10}{\log 3}$$

now $3^6 = 729$, so

$$\Leftrightarrow 7.29 > \frac{\log 10}{\log 3}$$

Taking log as $\log_3$, RHS $= \log_3 10 < 3$. So the inequality holds.

(iii) $\frac{a_7+1}{2a_7+1} = 0.50$, because a_7 has >100 digits, so $a_7+1 \approxeq a_7$, $2a_7+1 \approxeq 2a_7$, so

$$\frac{a_7+1}{2a_7+1} \approx \frac{a_7}{2a_7} = \frac{1}{2}.$$

STEP I 1997 Q4

We solve seperately for each region

$(-\infty, -1]$ $\quad f(x) = -x - 1 + x - 3(x-1) + 2(x-2) = -x - 2$

$[-1, 0]$ $\quad f(x) = x + 1 + x - 3(x-1) + 2(x-2) = x$

$[0, 1]$ $\quad f(x) = x + 1 - x - 3(x-1) + 2(x-2) = -x$

$[1, 2]$ $\quad f(x) = x + 1 - x + 3(x-1) + 2(x-2) = 5x - 6$

$[2, \infty)$ $\quad f(x) = x + 1 - x + 3(x-1) - 2(x-2) = x + 2$

$(-\infty, -1]$ $\quad -x - 2 = x + 2 \Rightarrow x = -2$ ✓ in range

$[-1, 0]$ $\quad x = x + 2$ no solution

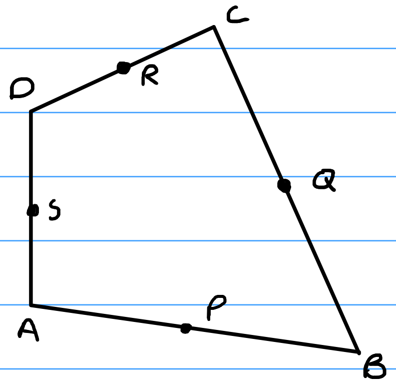
$[0, 1]$ $\quad -x = x + 2 \Rightarrow x = -1$ ✗ not in range

$[1, 2]$ $\quad 5x - 6 = x + 2 \Rightarrow x = 2$ ✓ in range

$[2, \infty)$ $\quad x + 2 = x + 2 \Rightarrow x \in \mathbb{R}$ ✓ for $x \geqslant 2$

So the solution is $x = -2$, or $x \geqslant 2$.

STEP I 1997 Q5



Throughout, we write e.g. $\underline{a}\cdot\underline{a}$ as a^2

$$|AB|^2-|BC|^2+|CD|^2-|DA|^2$$
$$=(\underline{b}-\underline{a})^2-(\underline{c}-\underline{b})^2+(\underline{d}-\underline{c})^2-(\underline{a}-\underline{d})^2$$
$$=\underline{b}^2-2\underline{a}\cdot\underline{b}+\underline{a}^2-\underline{c}^2+2\underline{b}\cdot\underline{c}-\underline{b}^2+\underline{d}^2-2\underline{c}\cdot\underline{d}+\underline{c}^2-\underline{a}^2+2\underline{a}\cdot\underline{d}-\underline{d}^2$$
$$=2(-\underline{a}\cdot\underline{b}+\underline{b}\cdot\underline{c}-\underline{c}\cdot\underline{d}+\underline{a}\cdot\underline{d})$$

Now, $\overrightarrow{QS}=\frac{1}{2}\overrightarrow{CB}+\overrightarrow{BA}+\frac{1}{2}\overrightarrow{AD}$
$$=\tfrac{1}{2}(-\underline{c}+\underline{b})+(-\underline{b}+\underline{a})+\tfrac{1}{2}(-\underline{a}+\underline{d})$$
$$=\tfrac{1}{2}(\underline{a}-\underline{b}-\underline{c}+\underline{d})$$

Similarly $\overrightarrow{PR}=\frac{1}{2}\overrightarrow{BA}+\overrightarrow{AD}+\frac{1}{2}\overrightarrow{DC}$
$$=\tfrac{1}{2}(-\underline{b}+\underline{a})+(-\underline{a}+\underline{d})+\tfrac{1}{2}(-\underline{d}+\underline{c})$$
$$=\tfrac{1}{2}(-\underline{a}-\underline{b}+\underline{c}+\underline{d})$$

So $2|QS|^2-2|PR|^2$
$$=\tfrac{1}{2}\left[(\underline{a}-\underline{b}-\underline{c}+\underline{d})^2-(-\underline{a}-\underline{b}+\underline{c}+\underline{d})^2\right]$$
$$=\tfrac{1}{2}\Big[\underline{a}^2+\underline{b}^2+\underline{c}^2+\underline{d}^2-2\underline{a}\cdot b-2\underline{a}\cdot c+2\underline{a}\cdot\underline{d}+2\underline{b}\cdot\underline{c}-2\underline{b}\cdot\underline{d}-2\underline{c}\cdot\underline{d}$$
$$-\underline{a}^2-\underline{b}^2-\underline{c}^2-\underline{d}^2-2\underline{a}\cdot\underline{b}+2\underline{a}\cdot\underline{c}+2\underline{a}\cdot\underline{d}+2\underline{b}\cdot\underline{c}+2\underline{b}\cdot\underline{d}+2\underline{c}\cdot\underline{d}\Big]$$
$$=2[-\underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{d}+\underline{b}\cdot\underline{c}-\underline{c}\cdot\underline{d}]$$
$$=|AB|^2-|BC|^2+|CD|^2-|DA|^2,\text{ as required.}$$

Now clearly $|AB|^2 - |BC|^2 + |CD|^2 - |DA|^2$ is constant, so
$|PR|^2 - |QS|^2 = -\frac{1}{2}\left(|AB|^2 - |BC|^2 + |CD|^2 - |DA|^2\right)$ is also constant.

$|AC||BD|\cos\theta$
$= \overrightarrow{AC} \cdot \overrightarrow{BD}$
$= (\underline{c} - \underline{a}) \cdot (\underline{d} - \underline{b})$
$= \underline{c} \cdot \underline{d} - \underline{b} \cdot \underline{c} - \underline{a} \cdot \underline{d} + \underline{a} \cdot \underline{b}$
$= \frac{1}{2}\left(|AB|^2 - |BC|^2 + |CD|^2 - |DA|^2\right)$, which is constant, as required.

STEP I 1997 Q6

$$x^4(1-x)^4 = x^4(1-4x+6x-4x^3+x^4)$$
$$= x^8-4x^7+6x^6-4x^5+x^4$$

$$
\begin{array}{r}
x^6-4x^5+5x^4+0x^3-4x^2+0x+4 \\
x^2+1 \,\big)\, \overline{x^8-4x^7+6x^6-4x^5+x^4+0x^3+0x^2+0x+0} \\
\underline{x^8+0x^7+x^6} \qquad\qquad\qquad\qquad\qquad\qquad \\
-4x^7+5x^6-4x^5 \qquad\qquad\qquad\qquad\qquad \\
\underline{-4x^7+0x^6-4x^5} \qquad\qquad\qquad\qquad\qquad \\
5x^6+0x^5+x^4 \qquad\qquad\qquad\qquad \\
\underline{5x^6+0x^5+5x^4} \qquad\qquad\qquad\qquad \\
0x^5-4x^4+0x^3 \qquad\qquad\qquad \\
\underline{0x^5+0x^4+0x^3} \qquad\qquad\qquad \\
-4x^4+0x^3+0x^2 \qquad\qquad \\
\underline{-4x^4+0x^3-4x^2} \qquad\qquad \\
0x^3+4x^2+0x \qquad \\
\underline{0x^3+0x^2+0x} \qquad \\
4x^2+0x+0 \\
\underline{4x^2+0x+4} \\
-4
\end{array}
$$

So $x^4(1-x)^4 = (x^6-4x^5+5x^4-4x^2+4)(x^2+1)-4$

So $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \int_0^1 x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\,dx$

$$= \left[x^7/7 - 4x^6/6 + x^5 - 4x^3/3 + 4x - 4\arctan x\right]_0^1$$
$$= 1/7 - 2/3 + 1 - 4/3 - \pi$$
$$= 22/7 - \pi.$$

$$\int_0^1 x^4(1-x)^4\,dx = \int_0^1 x^8 - 4x^7 + 6x^6 - 4x^5 + x^4\,dx$$

$$= \left[\tfrac{1}{9}x^9 - \tfrac{1}{2}x^8 + \tfrac{6}{7}x^7 - \tfrac{2}{3}x^6 + \tfrac{1}{5}x^5\right]_0^1$$

$$= \frac{1}{630}$$

Now $\dfrac{x^4(1-x)^4}{1+x^2} > 0$ for $x \in [0,1]$, so $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx > 0$

$\Rightarrow 22/7 - \pi > 0$

$\Rightarrow \pi < 22/7$

And $\dfrac{x^4(1-x)^4}{1+x^2} - x^4(1-x)^4$

$= x^4(1-x)^4\left(\dfrac{1}{1+x^2} - 1\right)$

$= x^4(1-x)^4\left(\dfrac{-x^2}{1+x^2}\right) < 0$ for $x \in [0,1]$.

So, $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2} - x^4(1-x)^4\,dx < 0$

$\Rightarrow \dfrac{22}{7} - \pi - \dfrac{1}{630} < 0$

$\Rightarrow \pi > \dfrac{22}{7} - \dfrac{1}{630}$

So $\dfrac{22}{7} > \pi > \dfrac{22}{7} - \dfrac{1}{630}$.

STEP I 1997 Q7

Suppose $P(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$, then

$$\int_{-1}^{1} P(t)\,dt = \left[\alpha\frac{x^4}{4} + \beta\frac{x^3}{3} + \gamma\frac{x^2}{2} + \delta x\right]_{-1}^{1}$$

$$= \tfrac{2}{3}\beta + 2\delta.$$

Now, we want $u_2 = -u_1$ so that the α and γ cancel. The symmetry also suggests trying $a_1 = a_2$.

So, $a(P(u) + P(-u))$
$= 2a(\beta u^2 + \delta)$

So want $2au^2 = \frac{2}{3}$, and $2a = 2$
$\Rightarrow a = 1,\ u = 1/\sqrt{3}$

so $\int_{-1}^{1} P(t)\,dt = P(1/\sqrt{3}) + P(-1/\sqrt{3})$

STEP I 1997 Q8

$$x = a^x$$
$$\Leftrightarrow \ln x = x\ln a$$
$$\Leftrightarrow \ln x - x\ln a = 0$$

Now $\frac{d}{dx}(\ln x - x\ln a) = \frac{1}{x} - \ln a = 0$

$$\Rightarrow x = \frac{1}{\ln a}$$

Further, $\frac{d^2}{dx^2}(\ln x - x\ln a) = \frac{-1}{x^2} < 0$, so the stationary point is a maximum.

Further, when $x = \frac{1}{\ln a}$, $\ln x - x\ln a$

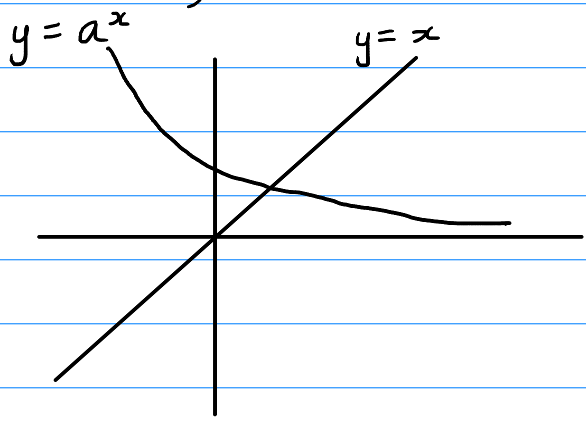
$$= \ln\left(\frac{1}{\ln a}\right) - 1$$

Now $a > e^{1/e}$

$$\Rightarrow \ln a > 1/e$$
$$\Rightarrow \frac{1}{\ln a} < e$$
$$\Rightarrow \ln\left(\frac{1}{\ln a}\right) < 1$$
$$\Rightarrow \ln\left(\frac{1}{\ln a}\right) - 1 < 0.$$

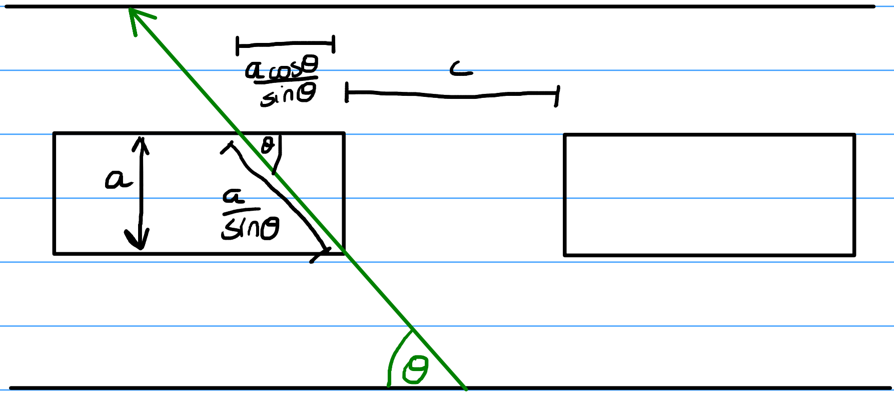
So, the maximum value of the function is negative, and the function is continuous (on its domain $x > 0$), and so it has no real roots.

For $0 < a < 1$,



Clearly there is one real root.

STEP I 1997 Q8



We want to just miss the back left corner of the first car, and cross just before we are hit by the front right corner of the next car.

We travel a distance of $a/\sin\theta$ in the path of traffic, which takes a time of $\frac{a}{u\sin\theta}$. We also travel left a distance of $\frac{a\cos\theta}{\sin\theta}$ in that time. Thus it takes the next car $\frac{1}{V}\left(c + \frac{a\cos\theta}{\sin\theta}\right)$ to reach us.

So, we require
$$\frac{a}{u\sin\theta} \leq \frac{1}{V}\left(c + \frac{a\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad \frac{aV}{u} \leq c\sin\theta + a\cos\theta$$

$$\Rightarrow \quad u \geq \frac{Va}{c\sin\theta + a\cos\theta}$$

The least possible speed is where equality holds, and where $\frac{Va}{c\sin\theta + a\cos\theta}$ is minimised as a function of θ.

This is when $c\sin\theta + a\cos\theta$ is maximised. The maximum value is $\sqrt{a^2+c^2}$, when $\frac{d}{d\theta} c\sin\theta + a\cos\theta = 0$

$$\Rightarrow \quad c\cos\theta - a\sin\theta = 0$$

$$\Rightarrow \quad \tan\theta = \frac{c}{a}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{c}{a}$$

$$\Rightarrow \frac{\sin^2\theta}{1-\sin^2\theta} = \frac{c^2}{a^2}$$

$$\Rightarrow \frac{c^2}{a^2}(1-\sin^2\theta) = \sin^2\theta$$

$$\Rightarrow \sin^2\theta(1+c^2/a^2) = c^2/a^2$$

$$\Rightarrow \sin^2\theta = \frac{c^2}{a^2} \cdot \frac{1}{1+c^2/a^2}$$

$$= \frac{c^2}{a^2+c^2}$$

So $\sin\theta = \frac{c}{\sqrt{a^2+c^2}}$

Now, the total distance travelled is $\frac{b}{\sin\theta}$, which takes

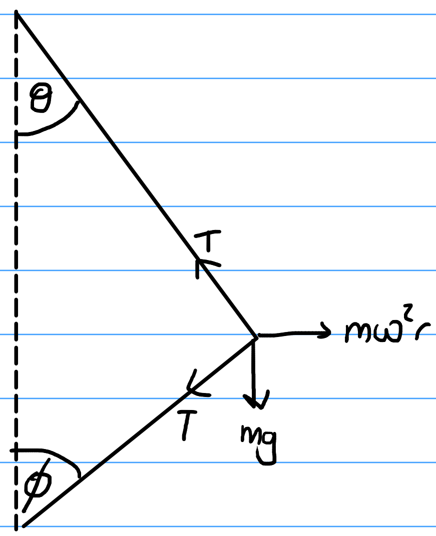
$$\frac{b}{\sin\theta} \cdot \frac{\sqrt{a^2+c^2}}{Va}$$

$$= \frac{b}{V} \cdot \frac{\sqrt{a^2+c^2}}{a} \cdot \frac{\sqrt{a^2+c^2}}{c}$$

$$= \frac{b}{V}\left(\frac{a^2+c^2}{ac}\right)$$

$$= \frac{b}{V}\left(\frac{a}{c}+\frac{c}{a}\right)$$

STEP I 1997 Q10



Resolving horizontally,

$$T\sin\theta + T\sin\phi = m\omega^2 r$$
$$\Rightarrow T(\sin\theta + \sin\phi) = m\omega^2 r$$
$$\Rightarrow 2T\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = m\omega^2 r \qquad (1)$$

Resolving vertically,

$$T\cos\theta = T\cos\phi + mg$$
$$\Rightarrow T(\cos\theta - \cos\phi) = mg$$
$$\Rightarrow -2T\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2} = mg \qquad (2)$$

$$(2) \div (1) \Rightarrow -\tan\frac{\theta-\phi}{2} = \frac{g}{\omega^2 r}$$

$$\Rightarrow \tan\frac{\phi-\theta}{2} = \frac{g}{\omega^2 r}$$

Then ① ^{2} + ② ^{2}

$$\Rightarrow 4T^2\sin^2\frac{\theta+\phi}{2}\cos^2\frac{\theta-\phi}{2} + 4T^2\sin^2\frac{\theta+\phi}{2}\sin^2\frac{\theta-\phi}{2} = m^2(g^2+r^2\omega^4)$$

$$\Rightarrow 4T^2\sin^2\frac{\theta+\phi}{2}\left(\sin^2\frac{\theta-\phi}{2} + \cos^2\frac{\theta-\phi}{2}\right) = m^2(g^2+r^2\omega^4)$$

$$\Rightarrow T = \frac{m}{2\sin\frac{\theta+\phi}{2}}\sqrt{g^2+r^2\omega^4}.$$

Now, if $r\omega^2 \ll g$, then $\tan\frac{\phi-\theta}{2} = \frac{g}{\omega^2 r} \gg 1 \Rightarrow \frac{\phi-\theta}{2} \approx \pi/2$

$$\Rightarrow \phi-\theta \approx \pi$$

But $0 \leq \theta, \phi \leq \pi$, so $\theta \approx 0$, $\phi \approx \pi$

Then $\sin\frac{\theta+\phi}{2} \approx \sin\pi/2 = 1$

So $T \approx \frac{m}{2}\sqrt{g^2}$
$= \frac{mg}{2}$

STEP I 1997 Q11

Consider conservation of energy.

Then $\frac{1}{2}u^2 = \frac{1}{2}v^2 + gs + \int_0^s kv^2\, ds$

Considering v^2 as a function of s, we differentiate w.r.t s:

$$0 = \frac{1}{2}\frac{d}{ds}(v^2) + g + kv^2$$
$$\Rightarrow \frac{d}{ds}(v^2) + 2kv^2 = -2g$$

IF $= e^{2ks}$, so

$$\frac{d}{ds}(v^2 e^{2ks}) = -2ge^{2ks}$$
$$\Rightarrow v^2 e^{2ks} = -\frac{g}{k}e^{2ks} + A$$
$$\Rightarrow v^2 = Ae^{-2ks} - \frac{g}{k}$$

Now $v^2(0) = u^2 \Rightarrow u^2 = A - g/k$

$$\Rightarrow A = u^2 + g/k$$

So $v^2 = (u^2 + g/k)e^{-2ks} - g/k$

$= u^2 e^{-2ks} + g/k(e^{-2ks} - 1)$, as required.

The maximum height is when $v^2 = 0$, so

$$e^{-2ks}(u^2 + g/k) = g/k$$
$$\Rightarrow e^{-2ks} = \frac{g}{ku^2+g}$$
$$\Rightarrow e^{2ks} = \frac{ku^2+g}{g}$$
$$\Rightarrow s = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$$

(*) does not hold on the downward path, as now gravity acts with, rather than against, the motion of the particle. Instead, $v^2 = u^2 e^{-2ks} + g/k(e^{-2ks} + 1)$

STEP I 1997 Q12

$x^2 + 2x + T = 0$

$\Rightarrow X = \dfrac{-2+\sqrt{4-4T}}{2}$

$= -1 + \sqrt{1-T}$ where $T \sim u[0,1]$

so $P(X > -1/3) = P(-1 + \sqrt{1-T} > -1/3)$

$= P(\sqrt{1-T} > 2/3)$

$= P(1 - T > 4/9)$

$= P(T < 5/9)$

$= 5/9$

$EX = -1 + E\sqrt{1-T}$

$= -1 + \int_0^1 \sqrt{1-x}\,dx$

$= -1 + \int_0^1 \sqrt{x}\,dx$

$= -1 + 2/3$

$= -1/3$

$EX^2 = E(-1 + \sqrt{1-T})^2$

$= E(1 - 2\sqrt{1-T} + 1 - T)$

$= 2 - ET - 2E\sqrt{1-T}$

$= 2 - 1/2 - 4/3$

$= 1/6$

So $\text{Var}X = \frac{1}{6} - (-1/3)^2 = 1/18$.

Now $EY = 800EX = \dfrac{-800}{3}$

$\text{Var}Y = 800\text{Var}X = \dfrac{800}{18}$

By the central limit theorem, Y is approximately normal.

So we take $Y \sim N\left(-\frac{800}{3}, \frac{800}{18}\right)$.

By tables, $P(Z \leq -1.4051) = 0.08$

$$\Rightarrow \frac{k + \frac{800}{3}}{\sqrt{\frac{800}{18}}} = -1.4051$$

$$\Rightarrow k = -276$$

STEP I 1997 Q13

$$P(\text{bomb}\mid\text{no canary}) = \frac{P(\text{bomb}\cap\text{no canary})}{P(\text{no canary})}$$

$$= \frac{P(\text{bomb})}{P(\text{no canary})}$$

$$= \frac{\sum P(\text{bomb}\mid\text{Mr } i)\,P(\text{Mr } i)}{\sum P(\text{no canary}\mid\text{Mr } i)\,P(\text{Mr } i)}$$

$$= \frac{\sum \frac{i}{10}\cdot\frac{i}{10}}{\sum \frac{2i+1}{10}\cdot\frac{i}{10}}$$

$$= \frac{\sum i^2}{\sum i(2i+1)}$$

$$= \frac{1+4+9+16}{1\times3+2\times5+3\times7+4\times9}$$

$$= \frac{30}{70} = 3/7$$

So $P(\text{anaconda}\mid\text{no canary}) = 1 - 3/7 = 4/7$

Hence $E(\text{survival if turn on light}) = \frac{1}{2}\times\frac{3}{7} + \frac{2}{3}\times\frac{4}{7}$
$= 25/42$

$E(\text{survival if leave light off}) = \frac{3}{7} + \frac{1}{2}\times\frac{4}{7}$
$= 5/7$
$= 30/42$

So Mr Blond should leave the light off.

STEP I 1997 Q14

$$E\,\text{cost} = \int_0^y ye^{-x}\,dx + \int_y^{\infty}[(1-s)y + r + sx]e^{-x}\,dx$$

$$= y[-e^{-x}]_0^y + ((1-s)y+r)[-e^{-x}]_y^{\infty} + s[-xe^{-x}]_y^{\infty} + s\int_y^{\infty} e^{-x}\,dx$$

$$= y(1-e^{-y}) + e^{-y}(y - sy + r) + sye^{-y} + se^{-y}$$

$$= y - \cancel{ye^{-y}} + \cancel{ye^{-y}} - \cancel{sye^{-y}} + re^{-y} + \cancel{sye^{-y}} + se^{-y}$$

$$= y + (r+s)e^{-y}$$

So $\frac{d(\text{cost})}{dy} = 1 - e^{-y}(r+s) = 0$

$$\Rightarrow e^{y} = r+s$$

$$\Rightarrow y = \ln(r+s)$$

Further, $\frac{d^2(\text{cost})}{dy^2} = e^{-y}(r+s) > 0$, so this is a minimum point, and the function is increasing for $y > \ln(r+s)$.

Thus if $r+s<1$ so $\ln(r+s)<0$, the minimum point occurs at a negative value of y. But we must have $y \geq 0$, so take $y=0$ to minimise cost.