STEP III 1996 Q1

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

So $\cosh^4 x - \sinh^4 x$

$= \frac{1}{16}\left[(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}) - (e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x})\right]$

$= \frac{1}{16}(8e^{2x} + 8e^{-2x})$

$= \frac{1}{2}(e^{2x} + e^{-2x})$

$= \cosh 2x$, as required.

$\cosh^4 x + \sinh^4 x$

$= \frac{1}{16}\left[(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}) + (e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x})\right]$

$= \frac{1}{16}\left[2e^{4x} + 12 + 2e^{-4x}\right]$

$= \frac{1}{4} \times \frac{1}{2}(e^{4x} + e^{-4x}) + \frac{3}{4}$

$= \frac{1}{4}\cosh 4x + \frac{3}{4}$, as required.

Now $\cosh^n x = \frac{1}{2^n}(e^x + e^{-x})^n$

$= \frac{1}{2^n}\left(e^{nx} + ne^{(n-2)x} + \binom{n}{2}e^{(n-4)x} + \_\_\_ + \binom{n}{2}e^{-(n-4)x} + ne^{-(n-2)x} + e^{-nx}\right)$

$= \frac{1}{2^{n-1}}\left(\cosh nx + n\cosh(n-2)x + \binom{n}{2}\cosh(n-4)x + \_\_\_ + \begin{cases} \binom{n}{\frac{n-1}{2}}\cosh x & n \text{ odd} \\ \frac{1}{2}\binom{n}{n/2} & n \text{ even} \end{cases}\right)$

$$\text{So } \cosh^n x = \begin{cases} \frac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{\frac{n-1}{2}-i}\cosh(2i+1)x & n \text{ odd} \\ \frac{1}{2^n}\binom{n}{n/2} + \frac{1}{2^{n-1}}\sum_{i=1}^{n/2}\binom{n}{\frac{n}{2}-i}\cosh 2ix & n \text{ even} \end{cases}$$

Hence $\cosh^{2m} x = \frac{1}{2^{2m}}\binom{2m}{m} + \frac{1}{2^{2m-1}}\sum_{i=1}^{m}\binom{2m}{m-i}\cosh 2ix$

The expansion of $\sinh^{2m}x$ will have alternate terms negative, so

$$\sinh^{2m}x = (-1)^m \frac{1}{2^{2m}}\binom{2m}{m} + \frac{1}{2^{2m-1}}\sum_{i=1}^{m}(-1)^{m+i}\binom{2m}{m-i}\cosh 2ix$$

So
$$\cosh^{2m}x - \sinh^{2m}x = \left(1-(-1)^m\right)\frac{1}{2^{2m}}\binom{2m}{m} + \frac{1}{2^{2m-1}}\sum_{i=1}^{m}\left(1-(-1)^{m+i}\right)\binom{2m}{m-i}\cosh 2ix$$

and
$$\cosh^{2m}x + \sinh^{2m}x = \left(1+(-1)^m\right)\frac{1}{2^{2m}}\binom{2m}{m} + \frac{1}{2^{2m-1}}\sum_{i=1}^{m}\left(1+(-1)^{m+i}\right)\binom{2m}{m-i}\cosh 2ix$$

STEP III 1996 Q2

We have $\begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2b \end{pmatrix}$

Now $\begin{vmatrix} 1 & 1 & a \\ 1 & a & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + a\begin{vmatrix} 1 & a \\ 2 & 1 \end{vmatrix}$

$$= (a-1) - (1-2) + a(1-2a)$$
$$= 2a - 2a^2 = 0$$
$$\Rightarrow a = 0, 1$$

So the matrix is invertible for $a \neq 0, 1$. Now we invert the matrix.

$$M = \begin{pmatrix} a-1 & -1 & 1-2a \\ 1-a & 1-2a & -1 \\ 1-a^2 & 1-a & a-1 \end{pmatrix}$$

$$C = \begin{pmatrix} a-1 & 1 & 1-2a \\ a-1 & 1-2a & 1 \\ 1-a^2 & a-1 & a-1 \end{pmatrix}$$

$$C^T = \begin{pmatrix} a-1 & a-1 & 1-a^2 \\ 1 & 1-2a & a-1 \\ 1-2a & 1 & a-1 \end{pmatrix}$$

So inverse is $\frac{1}{2a(1-a)} C^T$

$$= \frac{1}{2a}\begin{pmatrix} -1 & -1 & 1+a \\ \frac{1}{1-a} & \frac{1-2a}{1-a} & -1 \\ \frac{1-2a}{1-a} & \frac{1}{1-a} & -1 \end{pmatrix}$$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2a}\begin{pmatrix} -1 & -1 & 1+a \\ \frac{1}{1-a} & \frac{1-2a}{1-a} & -1 \\ \frac{1-2a}{1-a} & \frac{1}{1-a} & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 2b \end{pmatrix}$

$$= \frac{1}{a}\begin{pmatrix} ab+b-2 \\ 2-b \\ 2-b \end{pmatrix}$$

So for $a \neq 0, 1$, we have $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{a}\begin{pmatrix} ab+b-2 \\ 2-b \\ 2-b \end{pmatrix}$

If $a=0$, we have

$x+y=2$ ①

$x+z=2$ ②

$2x+y+z=2b$ ③

① and ② $\Rightarrow y=z$, then ③ becomes $2x+2y=2b$

$\Rightarrow y=b-x$

So for $a=0$, we have $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ b-k \\ b-k \end{pmatrix}$ for $k \in \mathbb{R}$.

If $a=1$, we have

$x+y+z=2$ ①

$x+y+z=2$ ②

$2x+y+z=2b$ ③

③ − ① $\Rightarrow x=2b-2$

Then ① $\Rightarrow 2b-2+y+z=2$

$\Rightarrow z=4-2b-y$

So for $a=1$, we have $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2b-2 \\ k \\ 4-2b-k \end{pmatrix}$ for $k \in \mathbb{R}$.

STEP III 1996 Q3

We want to find $I = \int_0^\theta \frac{1}{1-a\cos x}\,dx$

Set $t = \tan\frac{1}{2}x$, so $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}t = \frac{1}{2}(1+t^2)$, and $\cos x = \frac{1-t^2}{1+t^2}$.

So $$I = \int_0^{\tan\frac{1}{2}\theta} \frac{1}{1-a\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int_0^{\tan\frac{1}{2}\theta} \frac{2}{1+t^2-a(1-t^2)}\,dt$$

$$= \int_0^{\tan\frac{1}{2}\theta} \frac{2}{(1-a)+(1+a)t^2}\,dt$$

$$= \frac{2}{1+a}\int_0^{\tan\frac{1}{2}\theta} \frac{1}{\frac{1-a}{1+a}+t^2}\,dt$$

$$= \left[\frac{2}{1+a}\sqrt{\frac{1+a}{1-a}}\arctan\left(t\sqrt{\frac{1+a}{1-a}}\right)\right]_0^{\tan\frac{1}{2}\theta}$$

$$= \frac{2}{\sqrt{(1+a)(1-a)}}\arctan\left(\sqrt{\frac{1+a}{1-a}}\tan\frac{1}{2}\theta\right)$$

Now to find $\int_0^{\pi/2} \frac{1}{2-a\cos x} = \frac{1}{2}\int_0^{\pi/2}\frac{1}{1-\frac{a}{2}\cos x}\,dx$, we set $\theta = \pi/2$ and $a = a/2$

$$= \frac{1}{2}\cdot\frac{2}{\sqrt{1-a^2/4}}\arctan\left(\sqrt{\frac{1+a/2}{1-a/2}}\tan\pi/4\right)$$

$$= \frac{2}{\sqrt{4-a^2}}\arctan\sqrt{\frac{2+a}{2-a}}$$, as required.

Now want $\int_0^{3\pi/4}\frac{1}{\sqrt{2}+\cos x}\,dx = \frac{1}{\sqrt{2}}\int_0^{3\pi/4}\frac{1}{1+\frac{\sqrt{2}}{2}\cos x}\,dx$, so set $\theta = \frac{3\pi}{4}$ and $a = -\sqrt{2}/2$.

$$= \frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{1-1/2}}\arctan\left(\sqrt{\frac{1-\sqrt{2}/2}{1+\sqrt{2}/2}}\tan\frac{3\pi}{8}\right)$$

$$= 2\arctan\left(\sqrt{\frac{(1-\sqrt{2}/2)(1+\sqrt{2}/2)}{(1+\sqrt{2}/2)^2}}\tan\frac{3\pi}{8}\right)$$

$$= 2\arctan\left(\frac{1}{1+\sqrt{2}/2}\sqrt{1-1/2}\tan\frac{3\pi}{8}\right)$$

$$= 2\arctan\left(\frac{2}{2+\sqrt{2}}\cdot\frac{\sqrt{2}}{2}\tan\frac{3\pi}{8}\right)$$

$$= 2\arctan\left(\frac{\sqrt{2}}{2+\sqrt{2}}\tan\frac{3\pi}{8}\right)$$

$$= 2\arctan\left(\frac{1}{\sqrt{2}+1}\tan\frac{3\pi}{8}\right)$$

Now $-1 = \tan\frac{3\pi}{4} = \dfrac{2\tan\frac{3\pi}{8}}{1-\tan^2\frac{3\pi}{8}}$

$\Rightarrow \tan^2\frac{3\pi}{8} - 2\tan\frac{3\pi}{8} - 1 = 0 \Rightarrow \tan\frac{3\pi}{8} = 1 \pm \sqrt{2}$. But $\tan\frac{3\pi}{8} > 0$ so $\tan\frac{3\pi}{8} = 1+\sqrt{2}$.

So the integral is $2\arctan\left(\frac{1}{\sqrt{2}+1}(1+\sqrt{2})\right)$

$= 2\arctan 1$

$= 2\cdot\frac{\pi}{4}$

$= \pi/2$, as required.

STEP III 1996 Q4

$k \leq 2(k-2)$

$\Rightarrow k \leq 2k-4$

$\Rightarrow k \geqslant 4$

Note that including a 1 in the sum is pointless, as it doesn't increase the product. So, we consider possible sums for each number without any 1s.

5

5	5	
2 3	6 ✓	$P(5)=6$

6

6	6	
2 4	8	
3 3	9 ✓	$P(6)=9$
2 2 2	8	

7

7	7	
2 2 3	12 ✓	$P(7)=12$
2 5	10	
3 4	12	

8

8	8	
2 2 2 2	16	
2 2 4	16	
2 3 3	18 ✓	$P(8)=18$
3 5	15	
4 4	16	

9

9	9	
2 2 2 3	24	
2 2 5	20	
2 3 4	24	
2 7	14	
3 3 3	27 ✓	$P(9)=27$
3 6	18	

Now consider $P(1000)$. From the first inequality for $k \geqslant 4$, we are better off replacing k with 2 and $k-2$ (or at least no worse off). So the sum will only contain 2s and 3s. Further, $2+2+2=3\times3$, but $2\times2\times2<3\times3$, so we want to include as many 3s as possible.

$1000/6 = 166r4$, so $P(1000) = 3^{2\times166}\times2^2$

$= 2^2\times3^{332}$.

STEP III 1996 Q5

$\cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$

$$= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta$$
$$+ 35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$$

Now writing $\cos\theta = c$, and $\sin\theta = s$, we have

$\cos 7\theta = c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6$
$\sin 7\theta = 7c^6s - 35c^4s^3 + 21c^2s^5 - s^7$

So $\tan 7\theta = \dfrac{7c^6s - 35c^4s^3 + 21c^2s^5 - s^7}{c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6}$

Dividing top and bottom by c^7, we get

$$\tan 7\theta = \frac{7t - 35t^3 + 21t^5 - t^7}{1 - 21t^2 + 35t^4 - 7t^6}$$
$$= \frac{t(t^6 - 21t^4 + 35t^2 - 7)}{7t^6 - 35t^4 + 21t^2 - 1}$$, as required.

Now $\tan 7\theta = 0 \Rightarrow \theta = 0, \pm \pi/7, \pm 2\pi/7, \pm 3\pi/7$
$\Rightarrow \tan\theta = 0, \pm\tan \pi/7, \pm\tan\frac{2\pi}{7}, \pm\tan\frac{3\pi}{7}$

So these are the roots of $t(t^6 - 21t^4 + 35t^2 - 7) = 0$. Hence setting $x = t^2$, the roots of $x^3 - 21x^2 + 35x - 7$ are $\tan^2\frac{\pi}{7}, \tan^2\frac{2\pi}{7}, \tan^2\frac{3\pi}{7}$. Hence $\tan^2\frac{\pi}{7}\tan^2\frac{2\pi}{7}\tan^2\frac{3\pi}{7} = 7$, so $\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7} = \sqrt{7}$.

Now note that $\tan 7\theta$ is undefined for $\theta = \pm\frac{\pi}{14}, \pm\frac{3\pi}{14}, \pm\frac{5\pi}{14}$, so $\pm\tan\frac{\pi}{14}, \pm\tan\frac{3\pi}{14}, \pm\tan\frac{5\pi}{14}$ are the roots of $7t^6 - 35t^4 + 21t^2 - 1$. As before, setting $x = t^2$, the roots of $7x^3 - 35x^2 + 21x - 1$ are $\tan^2\frac{\pi}{14}, \tan^2\frac{3\pi}{14}, \tan^2\frac{5\pi}{14}$. Hence $\tan^2\frac{\pi}{14} + \tan^2\frac{3\pi}{14} + \tan^2\frac{5\pi}{14} = \frac{35}{7} = 5$.

STEP III 1996 Q6

(i) $\begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix} \in S$. So S is closed under matrix multiplication. Further $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$, so $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is the identity. Associativity follows from associativity of matrix multiplication. Also, $\begin{pmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, so inverses exist. Hence S is a group.

(ii) Suppose $AB = E$, with $\det A = 0$. Then

$$\det(AB) = \det E$$
$$\det A \det B = \det E$$
$$0 \times \det B = \det E$$

So $\det E = 0$.

Suppose $C \in G$. then $\det C$

$$= \det(EC)$$
$$= \det E \det C$$
$$= 0 \times \det C$$
$$= 0.$$

So if $\det A = 0$, all elements of G have determinant zero.

A group is matrices of the form $\begin{pmatrix} a & a & a \\ a & a & a \\ a & a & a \end{pmatrix}$ with the same reasoning as above, but with the identity as $\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and inverses as $\frac{1}{9a}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

STEP III 1996 Q7

(i) $\alpha^3 = (x+y+z)^3$
$= (x^3+y^3+z^3) + 3(x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2) + 6xyz$

$\alpha\beta = (x+y+z)(xy+yz+zx)$
$= (x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2) + 3xyz$

So $\alpha^3 = x^3+y^3+z^3 + 3(\alpha\beta - 3xyz) + 6xyz$
$= x^3+y^3+z^3 + 3\alpha\beta - 3\gamma$

Hence $x^3+y^3+z^3 = \alpha^3 - 3\alpha\beta + 3\gamma$.

Now consider $x+y+z=1$
$x^2+y^2+z^2 = 3$
$x^3+y^3+z^3 = 4$

Note $\alpha^2 = (x+y+z)^2 = x^2+y^2+z^2 + 2(xy+yz+xz)$
So $x^2+y^2+z^2 = \alpha^2 - 2\beta$.

So the equations become
$\alpha = 1$
$\alpha^2 - 2\beta = 3 \Rightarrow \beta = -1$
$\alpha^3 - 3\alpha\beta + \gamma = 4 \Rightarrow \gamma = 0$

So $x+y+z=1$
$xy+yz+xz = -1$
$xyz = 0$

These equations are symmetric in x, y, z, so wlog take $x=0$.

Then $y+z=1$

$yz=-1$

$\Rightarrow y(1-y)=-1$

$\Rightarrow y^2-y-1=0$

$\Rightarrow y=\frac{1\pm\sqrt{5}}{2},\ z=\frac{1\mp\sqrt{5}}{2}.$

So $\{x,y,z\}=\{0,\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\}$

(ii) $x^3-16x^2+81x-128=0$

So $abc=128$

$ab+bc+ac=81$

$a+b+c=16 \Rightarrow s=8$

So area $=\sqrt{s(s-a)(s-b)(s-c)}$

$=\sqrt{s(s^3-s^2(a+b+c)+s(ab+bc+ac)-abc)}$

$=\sqrt{8(512-64\times 16+8\times 81-128)}$

$=8$

STEP III 1996 Q8

We solve $T(x)=x$, so $\dfrac{ax-b}{cx-d}=x \Rightarrow ax-b=x(cx-d)$

$$\Rightarrow cx^2-(a+d)x+b=0$$

$$\Rightarrow x=\frac{a+d\pm\sqrt{(a+d)^2-4bc}}{2c}$$

Now suppose $\dfrac{ax-b}{cx-d}=y$

$$\Rightarrow ax-b=cyx-dy$$

$$\Rightarrow dy-b=x(cy-a)$$

$$\Rightarrow x=\frac{dy-b}{cy-a}$$

So $T^{-1}(x)=\dfrac{dx-b}{cx-a}$ with $a'=d$, $b'=b$, $c'=c$, $d'=a$.

For $c\neq 0$, T is defined for all $x\neq a/c$. Further the domain of the inverse is all $x\neq c/a$, and this is the range of T. So $T(S_r)=\mathbb{R}\setminus\{a/c\}$.

For $c=0$, $T(x)=\frac{-a}{d}x+\frac{b}{d}$ is defined for all real x. So $T(S_r)=\mathbb{R}\setminus\{\frac{-a}{d}r+\frac{b}{d}\}$.

$$T_3(x)=\frac{a_2\frac{a_1x-b_1}{c_1x-d_1}-b_2}{c_2\frac{a_1x-b_1}{c_1x-d_1}-d_2}$$

$$=\frac{a_2(a_1x-b_1)-b_2(c_1x-d_1)}{c_2(a_1x-b_1)-d_2(c_1x-d_1)}$$

$$=\frac{x(a_2a_1-b_2c_1)-(a_2b_1-b_2d_1)}{x(c_2a_1-d_2c_1)-(c_2b_1-d_1d_2)}$$ is of the required form.

For $T^2(x)=x$, we need (from the above)

$$a^2-bc=1$$
$$ab-bd=0$$
$$ac-dc=0$$
$$d^2-bc=1$$

$$\Rightarrow a^2=d^2.$$

If $a=d$, then the equations become $bc=0$.
If $a=-d$, then $b=c=0$ and so $a=-d=\pm1$.

So for $T^2(x)=0$ we have

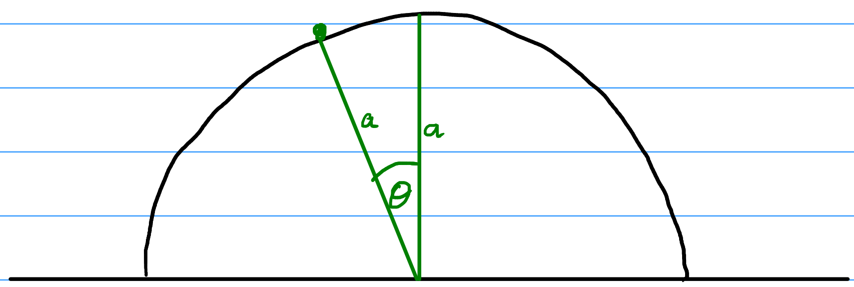
$$(a,b,c,d)=\begin{cases}(a,b,0,a)\\(a,0,c,a)\\(1,0,0,-1)\\(-1,0,0,1)\end{cases}$$

Then consider $T_1(x)=-x$ so $T_1^2(x)=x$ and $T_2(x)=1-x$ so $T_2^2(x)=x$. Then

$$\begin{aligned}T_3(x)&=T_2(T_1(x))\\&=1-(-x)\\&=1+x.\end{aligned}$$

Then
$$\begin{aligned}T_3^2(x)&=1+(1+x)\\&=2+x\neq x.\end{aligned}$$

STEP III 1996 Q9



When the particle has moved around the sphere by angle θ, it has lost $a(1-\cos\theta)$ of height. So using conservation of energy,

$$\frac{1}{2}mv^2 = mga(1-\cos\theta)$$
$$\Rightarrow v^2 = 2ga(1-\cos\theta)$$

The acceleration due to gravity tangential to the sphere is $g\cos\theta$. The centripetal acceleration is $\frac{v^2}{a}$. So the particle leaves the sphere when

$$g\cos\theta = \frac{v^2}{a}$$
$$\Rightarrow g\cos\theta = 2g(1-\cos\theta)$$
$$\Rightarrow \cos\theta = \frac{2}{3}$$
$$\Rightarrow \sin\theta = \frac{\sqrt{5}}{3}$$

At this point, the height of the particle above the plane is $\frac{2}{3}a$, and it has travelled $a\frac{\sqrt{5}}{3}$ horizontally. Further, its velocity is $v=\sqrt{\frac{2}{3}ag}$. The horizontal component is $v\cos\theta$ $=\sqrt{\frac{8}{27}ag}$, and the vertical component is $v\sin\theta = \sqrt{\frac{10}{27}ag}$.

s	$\frac{2}{3}a$
u	$\sqrt{\frac{10}{27}ag}$
v	$\times$
a	g
t	?

$$s = ut + \tfrac{1}{2}at^2$$
$$\Rightarrow \frac{2}{3}a = \sqrt{\frac{10}{27}ag}\,t + \frac{1}{2}gt^2$$
$$\Rightarrow \frac{1}{2}gt^2 + \sqrt{\frac{10}{27}ag}\,t - \frac{2}{3}a = 0$$
$$\Rightarrow t = \frac{-\sqrt{\frac{10}{27}ag} + \sqrt{\frac{10}{27}ag + \frac{4}{3}ag}}{g}$$

$$\Rightarrow t = \sqrt{\frac{a}{g}}\left(\sqrt{\frac{46}{27}} - \sqrt{\frac{10}{27}}\right)$$
$$= \sqrt{\frac{2a}{27g}}\left(\sqrt{23} - \sqrt{5}\right)$$

So between leaving the sphere and hitting the ground, the particle travels a further distance of

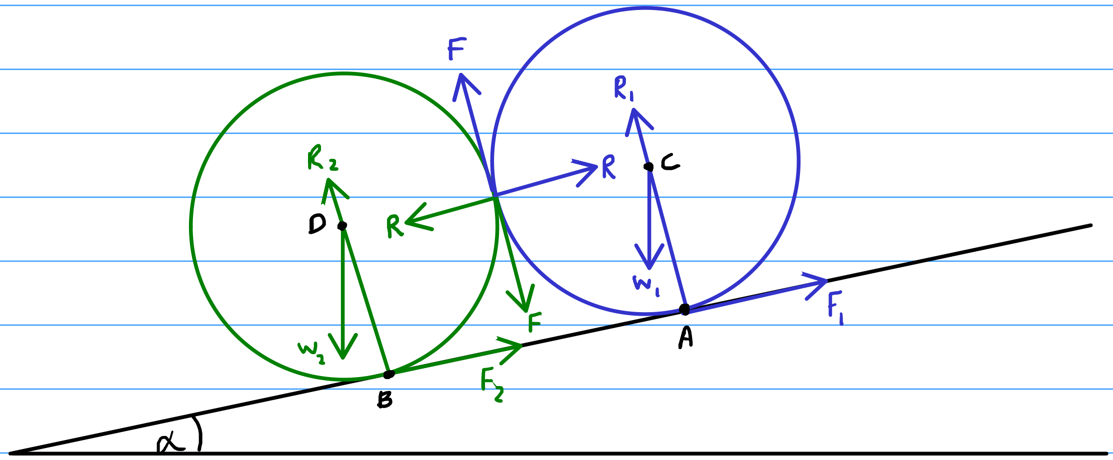
$$\sqrt{\frac{2a}{27g}}\left(\sqrt{23} - \sqrt{5}\right) \times \sqrt{\frac{8}{27}ag}$$

$$= \frac{4}{27}a\left(\sqrt{23} - \sqrt{5}\right).$$

So the total horizontal distance travelled is $\frac{4}{27}a\left(\sqrt{23} - \sqrt{5}\right) + a\frac{\sqrt{5}}{3}$

$$= \frac{a}{27}\left(5\sqrt{5} + 4\sqrt{23}\right)$$, as required.

STEP III 1996 Q9



$M(A) \Rightarrow W_1 \sin\alpha - R - F = 0 \qquad (1)$

$M(B) \Rightarrow W_2 \sin\alpha + R - F = 0 \qquad (2)$

$(2)-(1) \Rightarrow R = \frac{1}{2}(W_1 - W_2)\sin\alpha$

since $R > 0$ (and $\sin\alpha > 0$), we have $W_1 > W_2$. $\qquad (i)$

$(2)+(1) \Rightarrow F = \frac{1}{2}(W_1 + W_2)\sin\alpha$

Now $F \leq \mu R \Rightarrow \mu \geq \frac{F}{R} = \frac{W_1 + W_2}{W_1 - W_2} \qquad (ii)$

$M(D) \Rightarrow F = F_2$

$M(C) \Rightarrow F = F_1$. So $F_1 = F_2 = F$

$R(\updownarrow)$ For sphere 1: $W_1 \cos\alpha = R_1 + F$

$\Rightarrow R_1 = W_1 \cos\alpha - F$

But $F \leq \mu_1 R_1$

$\Rightarrow \mu_1 \geq \frac{F}{R_1}$

$= \left(\frac{R_1}{F}\right)^{-1}$

$$= \left(\frac{W_1\cos\alpha - F}{F}\right)^{-1}$$

$$= \left(\frac{W_1\cos\alpha}{F} - 1\right)^{-1}$$

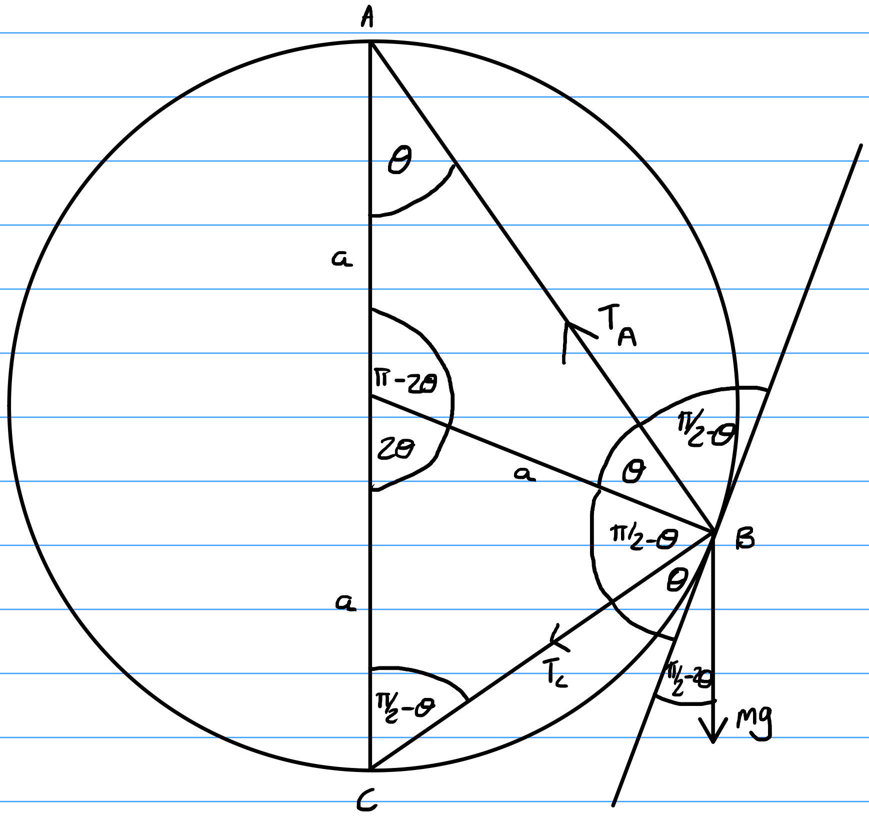
$$= \left(\frac{2W_1\cos\alpha}{(W_1+W_2)\sin\alpha} - 1\right)^{-1}$$

$$= \left(\frac{2W_1\cot\alpha}{W_1+W_2} - 1\right)^{-1} \qquad \text{(iii)}$$

For μ_2, the first line is $W_2\cos\alpha = R_2 - F$. The argument proceeds similarly to

$$\mu_2 \geq \left(\frac{2W_2\cot\alpha}{W_1+W_2} + 1\right)^{-1}$$

STEP III 1996 Q11



For the particle, resolving along the circumference, taking moving towards A the positive direction,

$$m\ddot{x} = T_A\cos(\pi/2-\theta) - T_c\cos\theta - mg\cos(\pi/2-2\theta)$$
$$= T_A\sin\theta - T_c\cos\theta - mg\sin 2\theta$$

But $x = a(2\theta)$

so $\ddot{x} = 2a\ddot{\theta}$

Further as $T = \frac{\lambda x}{l}$, we have $T_A = \lambda(2\cos\theta - 1)$ and $T_c = \lambda(2\sin\theta - 1)$

So $2ma\ddot{\theta} = \lambda\left((2\cos\theta - 1)\sin\theta - (2\sin\theta - 1)\cos\theta\right) - mg\sin 2\theta$
$= \lambda(\cos\theta - \sin\theta) - mg\sin 2\theta$, as required.

Now $\sin\theta = 3/5$, $\cos\theta = 4/5$, $\sin 2\theta = 24/25$

$\ddot{\theta} = 0 \Rightarrow \frac{1}{5}\lambda - \frac{24}{25}mg = 0$

$\Rightarrow 5\lambda = 24mg$, as required.

So $2ma\ddot{\theta} = \frac{24}{5}mg(\cos\theta - \sin\theta) - mg\sin 2\theta$

$\Rightarrow 10a\ddot{\theta} = 24g(\cos\theta - \sin\theta) - g\sin 2\theta$

considering a small pertubation $\theta + \varepsilon$,

$$10a\ddot{\varepsilon} = 24g(\cos(\theta+\varepsilon) - \sin(\theta+\varepsilon)) - 5g\sin(2(\theta+\varepsilon))$$
$$= 24g(\cos\theta\cos\varepsilon - \sin\theta\sin\varepsilon - \sin\theta\cos\varepsilon - \cos\theta\sin\varepsilon) - 5g(\sin 2\theta\cos 2\varepsilon + \cos 2\theta\sin 2\varepsilon)$$

Using $\sin\varepsilon \approx \varepsilon$, $\cos\varepsilon \approx 1$,

$$10a\ddot{\varepsilon} = 24g(\cos\theta - \varepsilon\sin\theta - \sin\theta - \varepsilon\cos\theta) - 5g(\sin 2\theta + 2\varepsilon\cos 2\theta)$$
$$= \underbrace{24g(\cos\theta - \sin\theta) - 5g\sin 2\theta}_{0} - \varepsilon g(24\cos\theta + 24\sin\theta + 10\cos 2\theta)$$
$$= - \varepsilon g\left(\frac{168}{5} + 10 \times \frac{7}{25}\right)$$
$$= -\varepsilon g \times \frac{182}{5}$$

So $\ddot{\varepsilon} = -\varepsilon \cdot \frac{g}{a} \cdot \frac{91}{25}$

So the oscillations have period $2\pi\sqrt{\frac{25a}{91g}}$

$$= 10\pi\sqrt{\frac{a}{91g}}.$$

STEP II 1996 Q12

(i) $b_j = P(\text{next is } j)$

$= \sum_i P(\text{next is } j \mid \text{current is } i) P(\text{current is } i)$

$= \sum_i p_{ij} a_i$

$= a_1 p_{1j} + a_2 p_{2j} + a_3 p_{3j}$

(ii) We have

$$(x \quad y \quad z)\begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{pmatrix} = (x \quad y \quad z)$$

So $-0.7x + 0.2y + 0.1z = 0$ ①

$0.3x - 0.6y + 0.3z = 0$ ②

$0.4x - 0.4y - 0.4z = 0$ ③

$3 \times ① + ② \Rightarrow -1.8x + 0.6z = 0 \Rightarrow z = 3x$

$2 \times ① - ③ \Rightarrow -1.8x + 0.6z = 0 \Rightarrow z = 3x$

Then substituting into ① $\Rightarrow -0.4x + 0.2y = 0 \Rightarrow y = 2x$.

Further $x+y+z=1 \Rightarrow (x \quad y \quad z) = (1/6 \quad 1/3 \quad 1/2)$

(iii) $q_{ij} = P(\text{last was } i \mid \text{current is } j)$

$= P(\text{current is } j \mid \text{last was } i) \dfrac{P(\text{last was } i)}{P(\text{current is } j)}$

$= p_{ij} \cdot \dfrac{a_i}{a_j}$

So $(Q)_{ij} = \begin{pmatrix} 3/10 & 3/20 & 2/5 \\ 2/5 & 2/5 & 4/15 \\ 3/10 & 9/20 & 3/5 \end{pmatrix}$

STEP III 1996 Q13

$$EX = \sum_x x P(X=x)$$

$$\text{Var}X = \sum_x (x-EX)^2 P(X=x)$$

$$E(aX+b) = \sum_x (ax+b)P(X=x)$$
$$= a\sum_x xP(X=x) + b\sum_x P(X=x)$$
$$= aEX+b \quad \text{(as probabilities sum to one).}$$

$$\text{Var}(aX+b) = \sum_x \left[ax+b - E(aX+b)\right]^2 P(X=x)$$
$$= \sum_x \left[ax+b-aEX-b\right]^2 P(X=x)$$
$$= \sum_x a^2(x-EX)^2 P(X=x)$$
$$= a^2 \text{Var}X$$

Suppose $|x_i - EX_i| \geqslant \lambda$ for all $i \in S$ (and $|x_i - EX_i| < \lambda$ for all $i \in S'$).

Then
$$\text{Var}X = \sum_x (x-EX)^2 P(X=x)$$
$$\geqslant \sum_{i\in S} (x_i - EX)^2 P(X=x_i)$$
$$\geqslant \lambda^2 \sum_{i\in S} P(X=x_i)$$

So
$$\frac{\text{Var}X}{\lambda^2} \geqslant \sum_{i\in S} P(X=x_i)$$
$$= P(|X-EX| \geqslant \lambda) \text{ by the definition of } S.$$

Suppose now $k \geqslant \lambda$, with $|x_i - EX| \leq k$ for all i, and S and S' as before.

Then
$$\text{Var}X \leq k^2 \sum_{i\in S} P(X=x_i) + \lambda^2 \sum_{i\in S'} P(X=x_i)$$

$$= \lambda^2 \sum_x P(X=x_i) + (k^2-\lambda^2)\sum_{i\in S} P(X=x_i)$$
$$= \lambda^2 + (k^2-\lambda^2)P\left[|X-EX| \geqslant \lambda\right]$$

So
$$\frac{\text{Var}X - \lambda^2}{k^2-\lambda^2} \leq P\left[|X-EX| \geqslant \lambda\right]$$

STEP III 1996 Q14

E(number of punctures) $= \int_0^T \alpha \, dt = \alpha T$.

$$p(t+\delta t) = \underbrace{p(t)(1-\beta\delta t)}_{\text{have a puncture already and don't fix it}} + \underbrace{(1-p(t))\alpha\delta t}_{\text{get a new puncture}} + O(\delta t^2)$$

So $\dfrac{p(t+\delta t)-p(t)}{\delta t} = \alpha - \alpha p(t) - \beta p(t)$

$\Rightarrow p'(t) = \alpha - (\alpha+\beta)p(t)$. $\quad (*)$

Setting $p(t) = \dfrac{\alpha}{\alpha+\beta}(1-e^{-(\alpha+\beta)t})$

$p'(t) = \alpha e^{-(\alpha+\beta)t}$

and $\alpha - (\alpha+\beta)p(t) = \alpha - \alpha(1-e^{-(\alpha+\beta)t})$

$= \alpha e^{-(\alpha+\beta)t}$, so p satisfies $(*)$.

Further, $p(0) = 0$, as needed.

E(time mending punctures) $= \int_0^T p(t)\,dt$

$$= \int_0^T \frac{\alpha}{\alpha+\beta}(1-e^{-(\alpha+\beta)t})\,dt$$

$$= \frac{\alpha}{\alpha+\beta}\left[t + \frac{e^{-(\alpha+\beta)t}}{\alpha+\beta}\right]_0^T$$

$$= \frac{\alpha}{\alpha+\beta}\left[T + \frac{1}{\alpha+\beta}(e^{-(\alpha+\beta)T}-1)\right]$$

So E(fraction of time fixing punctures)

$$= \frac{\alpha}{\alpha+\beta}\left[1 + \frac{1}{(\alpha+\beta)T}(e^{-(\alpha+\beta)T}-1)\right]$$

If $(\alpha+\beta)T$ is small, then $e^{-(\alpha+\beta)T} \approx 1-(\alpha+\beta)T+(\alpha+\beta)^2\frac{T^2}{2}$

so E(Fraction of time fixing punctures)

$$\approx \frac{\alpha}{\alpha+\beta}\left[1+\frac{1}{(\alpha+\beta)T}\left(1-(\alpha+\beta)T+(\alpha+\beta)^2\frac{T^2}{2}-1\right)\right]$$

$$=\frac{\alpha}{\alpha+\beta}\left[1-1+(\alpha+\beta)\frac{T}{2}\right]$$

$$=\frac{\alpha T}{2}$$

If $(\alpha+\beta)T$ is large, the second term in the bracket is small, so

E(Fraction of time fixing punctures) $\approx \frac{\alpha}{\alpha+\beta}$.