STEP II 1996 Q1

(i) We want the coefficient of x^6 from $(1-2x+3x^2-4x^3+5x^4)^3$

To make x^6, we could use

$1, 3x^2, 5x^4$ (6 ways)

$1, -4x^3, -4x^3$ (3 ways)

$-2x, -2x, 5x^4$ (3 ways)

$-2x, 3x^2, -4x^3$ (6 ways)

$3x^2, 3x^2, 3x^2$ (1 way)

So the coefficient is $6\times(1\times3\times5)+3\times(1\times-4\times-4)+3\times(-2\times-2\times5)+$
$6\times(-2\times3\times-4)+(3\times3\times3)$
$=90+48+60+144+27$
$=369$

(ii) Note $(1+x)^{-2}=(1-2x+3x^2-4x^3+5x^4-6x^5+7x^6+\dots)$

So the coefficient of x^6 in $(1-2x+3x^2-4x^3+5x^4-6x^5+7x^6)^3$ is the coefficient of x^6 in $[(1+x)^{-2}]^3$ (as all terms up to x^6 match), which is the coefficient of x^6 in $(1+x)^{-6}$.

this is $\dfrac{(-6)(-7)(-8)(-9)(-10)(-11)}{6!}=462$

STEP II 1996 Q2

Set $a = yz$, $b = zx$, $c = xy$, so

$2a + b - 5c = 2$ ①
$a - b + 2c = 1$ ②
$a - 2b + 6c = 3$ ③

② $\Rightarrow a = 1 + b - 2c$

Substituting into ①, $2 + 2b - 4c + b - 5c = 2 \Rightarrow 3b - 9c = 0 \Rightarrow b = 3c$

Substituting into ③, $1 + b - 2c - 2b + 6c = 3 \Rightarrow 4c - b = 2 \Rightarrow 4c - 3c = 2 \Rightarrow c = 2$

$b = 3c = 3 \times 2 = 6$

$a = 1 + b - 2c = 1 + 6 - 4 = 3$

So, $(xyz)^2 = abc = 2 \times 3 \times 6 = 36 \Rightarrow xyz = \pm 6$.

We have $yz = 3 \Rightarrow y = \frac{3}{z}$

$xz = 6 \Rightarrow x = \frac{6}{z}$

$xy = 2 \Rightarrow \frac{6}{z} \cdot \frac{3}{z} = 2 \Rightarrow z^2 = 9 \Rightarrow z = \pm 3$

If $z = 3$, then $y = 1$, $x = 2$

If $z = -3$, then $y = -1$, $x = -2$

So $(x, y, z) = (2, 1, 3)$ or $(-2, -1, -3)$.

STEP II 1996 Q3

$F_2 = 0+1 = 1$
$F_3 = 1+1 = 2$
$F_4 = 2+1 = 3$
$F_5 = 3+2 = 5$
$F_6 = 5+3 = 8$
$F_7 = 8+5 = 13$

$F_0 F_2 - F_1^2 = -1$
$F_1 F_3 - F_2^2 = 2 - 1^2 = 1$
$F_2 F_4 - F_3^2 = 3 - 2^2 = -1$
$F_3 F_5 - F_4^2 = 10 - 3^2 = 1$
So we hypothesise that $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$

Proof

The $n=1$ case is proved above.
Assume true for $n=k$, so $F_{k+1} F_{k-1} - F_k^2 = (-1)^k$

Then $F_{k+2} F_k - F_{k+1}^2$
$= (F_k + F_{k+1}) F_k - (F_k + F_{k-1}) F_{k+1}$
$= F_k^2 + F_k F_{k+1} - F_k F_{k+1} - F_{k-1} F_{k+1}$
$= -(F_{k+1} F_{k-1} - F_k^2)$
$= -(-1)^k$
$= (-1)^{k+1}$, as required.

True for $n=1$, and if true for $n=k$ then true for $n=k+1$, so true for all $n \in \mathbb{N}$.

Now, we want to show that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

We induct on k.

For $k=1$,

$$F_1 F_{n+1} + F_0 F_n$$
$$= F_{n+1} \checkmark$$

For $k=2$,

$$F_2 F_{n+1} + F_1 F_n$$
$$= F_{n+1} + F_n$$
$$= F_{n+2} \checkmark$$

Assume true for $k=\alpha-1$ and $k=\alpha$. Then for $k=\alpha+1$,

$$\begin{aligned} F_{n+\alpha+1} &= F_{n+\alpha} + F_{n+\alpha-1} && \text{(by Fibonacci property)} \\ &= F_\alpha F_{n+1} + F_{\alpha-1} F_n + F_{\alpha-1} F_{n+1} + F_{\alpha-2} F_n && \text{(by assumption)} \\ &= (F_\alpha + F_{\alpha-1}) F_{n+1} + (F_{\alpha-1} + F_{\alpha-2}) F_n \\ &= F_{\alpha+1} F_{n+1} + F_\alpha F_n \text{, as required} && \text{(by Fibonacci property)} \end{aligned}$$

True for $k=1$ and $k=2$, and if true for $k=\alpha-1$ and $k=\alpha$, then true for $k=\alpha+1$.
So true $\forall k \in \mathbb{N}$.

STEP II 1996 Q4

$$\cos 4u = 2\cos^2 2u - 1$$
$$= 2(2\cos^2 u - 1)^2 - 1$$
$$= 2(4\cos^4 u - 4\cos^2 u + 1) - 1$$
$$= 8\cos^4 u - 8\cos^2 u + 1$$

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2}\, dx \qquad x = \cos t,\ dx = -\sin t\, dt$$

$$= \int_{\pi}^{0} \frac{-\sin t\, dt}{\sqrt{1+\cos t} + \sqrt{1-\cos t} + 2}$$

$$\sqrt{1+\cos t} = \sqrt{1 + 2\cos^2\tfrac{t}{2} - 1}$$
$$= \sqrt{2}\cos t/2$$
$$\sqrt{1-\cos t} = \sqrt{1+2\sin^2 t - 1}$$
$$= \sqrt{2}\sin t$$

$$= \int_0^{\pi} \frac{\sin t\, dt}{\sqrt{2}(\sin t/2 + \cos t/2) + 2}$$

$$\sin t/2 + \cos t/2 = R\cos(t/2 + \alpha)$$
$$\Rightarrow R = \sqrt{2},\ \tan\alpha = -1 \Rightarrow \alpha = -\pi/4$$

$$= \int_0^{\pi} \frac{\sin t\, dt}{2\cos(t/2 - \pi/4) + 2}$$

$$2\cos(t/2 - \pi/4) = 2(2\cos^2(t/4 - \pi/8) - 1)$$
$$= 4\cos^2(t/4 - \pi/8) - 2$$

$$= \int_0^{\pi} \frac{\sin t\, dt}{4\cos^2(t/4 - \pi/8) - 2 + 2}$$

$$= \int_0^{\pi} \frac{\sin t}{4\cos^2(t/4 - \pi/8)}\, dt$$, as required.

Setting $u = t/4 - \pi/8$, $dt = 4du$

$$= \int_{-\pi/8}^{\pi/8} \frac{\sin(4u + \pi/2)}{\cos^2 u}\, du$$

$$= \int_{-\pi/8}^{\pi/8} \frac{\cos 4u}{\cos^2 u}\, du$$

$$= \int_{-\pi/8}^{\pi/8} \frac{8\cos^4 u - 8\cos^2 u + 1}{\cos^2 u}\, du$$

$$= \int_{-\pi/8}^{\pi/8} 8\cos^2 u - 8 + \sec^2 u \, du$$

$$= \int_{-\pi/8}^{\pi/8} 4\cos 2u - 4 + \sec^2 u \, du$$

$$= \Big[2\sin 2u - 4u + \tan u\Big]_{-\pi/8}^{\pi/8}$$

$$= \left(\sqrt{2} - \frac{\pi}{2} + \sqrt{2} - 1\right) - \left(-\sqrt{2} + \pi/2 - \sqrt{2} + 1\right)$$

$$= 4\sqrt{2} - \pi - 2$$, as required.

STEP II 1996 Q5

$z^4 + z^3 + z^2 + z + 1 = 0 \quad (*)$

$\Rightarrow z^2 + z + 1 + z^{-1} + z^{-2} = 0$ (and $z = 0$ is not a solution)

$\Rightarrow (z + z^{-1})^2 - 2 + (z + z^{-1}) + 1 = 0$

$\Rightarrow u^2 + u - 1 = 0$

$\Rightarrow u = \frac{-1 \pm \sqrt{5}}{2}$

So $z + \frac{1}{z} = \frac{-1+\sqrt{5}}{2}$ or $z + \frac{1}{z} = \frac{-1-\sqrt{5}}{2}$

$\Rightarrow z^2 + \left(\frac{1-\sqrt{5}}{2}\right)z + 1 = 0$ or $z^2 + \left(\frac{1+\sqrt{5}}{2}\right)z + 1 = 0$

$\Rightarrow z = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2} \pm \sqrt{\left(\frac{1-\sqrt{5}}{2}\right)^2 - 4}\right)$ or $z = \frac{1}{2}\left(\frac{-\sqrt{5}-1}{2} \pm \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2 - 4}\right)$

$\Rightarrow z = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2} \pm \frac{1}{2}\sqrt{1 - 2\sqrt{5} + 5 - 16}\right)$ or $z = \frac{1}{2}\left(\frac{-\sqrt{5}-1}{2} \pm \frac{1}{2}\sqrt{1 + 2\sqrt{5} + 5 - 16}\right)$

$\Rightarrow z = \frac{1}{4}\left(\sqrt{5} - 1 \pm i\sqrt{10 + 2\sqrt{5}}\right)$ or $z = \frac{1}{4}\left(-\sqrt{5} - 1 \pm i\sqrt{10 - 2\sqrt{5}}\right)$

Now, $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1) = 0$, so clearly if z satisfies $(*)$, then $z^5 - 1 = 0$. Thus the solutions of $(*)$ are the 5^{th} roots of unity (excluding $z = 1$).

So $z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, $z = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$, $z = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$, or $z = \frac{8\pi}{5} + i\sin\frac{8\pi}{5}$.

$\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is the only solution with both real and imaginary part positive, so we have $\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \frac{1}{4}\left(\sqrt{5} - 1 + i\sqrt{10 + 2\sqrt{5}}\right)$, so

$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$, $\sin\frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$, as required.

STEP II 1996 Q6

12 has factors 1, 2, 3, 4, 6, 12, so has 4 proper factors: 2, 3, 4, and 6.
16 has factors 1, 2, 4, 8, 16, so has 3 proper factors: 2, 4, and 8.

If $N = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$, then each factor of N is some combination of the p_i.
There are (m_1+1) choices for p_1 (can have $m_1 = 0$), and similar for the other p_i.

So N has $(m_1+1)(m_2+1)\dots(m_r+1)$ factors, but these include 1 and N, so N has $(m_1+1)(m_2+1)\dots(m_r+1)-2$ proper factors.

(i) We have $(m_1+1)(m_2+1)\dots(m_r+1) = 14$. Clearly we want $m_1 \geq m_2 \geq m_3 \dots$
But the only factor pairs of 14 are 14 and 1 or 7 and 2. So $N = 2^{13}$ or $N = 2^6 \times 3$.
$2^6 \times 3 = 192 < 2^{13}$, so $N = 192$.

(ii) Now $(m_1+1)(m_2+1)\dots(m_r+1) \geq 14$. As before, $m_1 \geq m_2 \geq m_3 \geq \dots$.

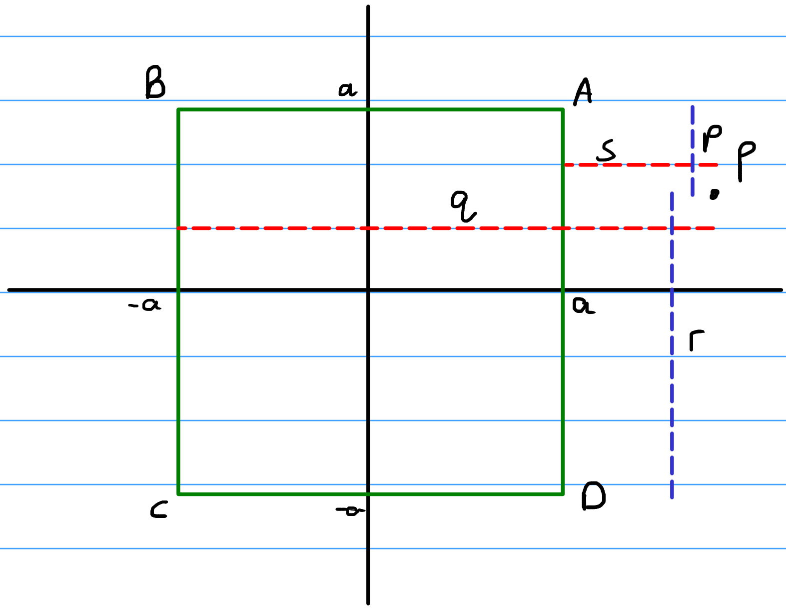
Note $m_1 = m_2 = m_3 = m_4 = 1$ works, then $N = 2 \times 3 \times 5 \times 7 = 210$. But from before $N = 2^6 \times 3$ also works, and so our final answer doesn't have any 7s. So, our options are

m_1+1	m_2+1	m_3+1	N
14	1	1	8192
7	2	1	192
5	3	1	144
5	2	1	720
4	4	1	216
4	3	1	360
4	2	2	120

So the smallest solution is $2^3 \times 3 \times 5 = 120$.

STEP II 1996 Q7

Suppose the square has vertices $A(a,a)$, $B(-a,a)$, $C(-a,-a)$, $D(a,-a)$, and P has coordinates (x,y). Then $pr = qs$



$$pr = qs \iff |x-a||x+a| = |y+a||y-a| \quad (*)$$
$$\iff ((x-a)(x+a))^2 = ((y+a)(y-a))^2$$
$$\iff (x^2-a^2)^2 = (y^2-a^2)^2$$
$$\iff x^4 - 2a^2x^2 + \cancel{a^4} = y^4 - 2a^2y^2 + \cancel{a^4}$$
$$\iff (x^4-y^4) - 2a^2(x^2-y^2) = 0$$
$$\iff (x^2-y^2)[(x^2+y^2) - 2a^2] = 0$$
$$\iff y^2 = x^2 \quad \text{or} \quad x^2+y^2 = 2a^2$$
$\iff y = \pm x$ or circle centred at the centre of the square with radius $\sqrt{2}a$.

Because each step is if and only if, the argument works in both directions.

STEP II 1996 Q8

We have $f''(x) + f(-x) = x + 3\cos 2x$, and $f(0) = 1$, $f'(0) = -1$.

$g(x) = f(x) + f(-x)$.
$g(0) = f(0) + f(0) = 2$.
$g'(x) = f'(x) - f'(-x)$.
$g'(0) = f(0) - f(0) = 0$.
$g''(x) = f''(x) + f''(-x)$.

Now, $f''(x) = x + 3\cos 2x - f(-x)$
$\Rightarrow f''(-x) = -x + 3\cos 2x - f(x)$

so $g''(x) = x + 3\cos 2x - f(-x) - x + 3\cos 2x - f(x)$.
$= 6\cos 2x - (f(x) + f(-x))$
$= 6\cos 2x - g(x)$.
So $g''(x) + g(x) = 6\cos 2x$.
CF: $g(x) = A\cos x + B\sin x$
PI: $g(x) = \lambda \cos 2x$
$\Rightarrow (-4\lambda + \lambda)\cos 2x = 6\cos 2x$
$\Rightarrow \lambda = -2$.

So $g(x) = A\cos x + B\sin x - 2\cos 2x$.
$g(0) = 2 \Rightarrow A = 4$
$g'(0) = 0 \Rightarrow B = 0$
So $g(x) = 4\cos x - 2\cos 2x$.

Now $h(x) = f(x) - f(-x)$

$h'(x) = f'(x) + f'(-x)$

$h''(x) = f''(x) - f''(-x)$

So $h(0) = 0$, $h'(0) = -2$

$h''(x) = f''(x) - f''(-x)$

$= 2x + f(x) - f(-x)$

$= 2x + h(x)$

So $h''(x) - h(x) = 2x$

CF: $h(x) = Ae^{x} + Be^{-x}$ PI: $h(x) = -2x$

So $h''(x) = Ae^{x} + Be^{-x} - 2x$

$h(0) = 0 \Rightarrow A + B = 0$

$h'(0) = -2 \Rightarrow A - B - 2 = -2$

$\Rightarrow A - B = 0$

$\Rightarrow A = B = 0.$

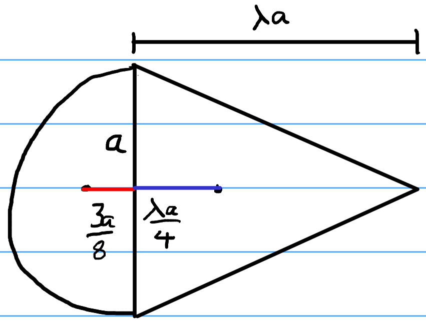
So $h(x) = -2x$

Now $f(x) = \frac{1}{2}(g(x) + h(x))$

$= \frac{1}{2}(4\cos x - 2\cos 2x - 2x)$

$= 2\cos x - \cos 2x - x$, as required.

STEP II 1996 Q9



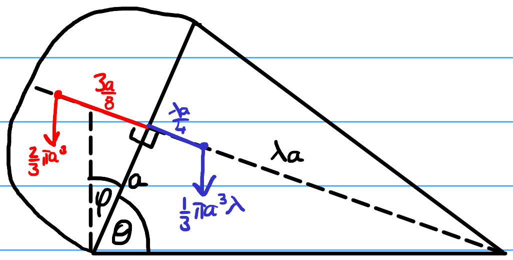
Overbalances if $\left(\frac{1}{3}\pi a^2 \lambda a\right) \times \frac{\lambda a}{4} > \left(\frac{2}{3}\pi a^3\right) \times \frac{3a}{8}$

$$\iff \frac{1}{12}\pi a^4 \lambda^2 > \frac{6}{24}\pi a^4$$

$$\iff \lambda^2 > 3$$

$$\Rightarrow \lambda > \sqrt{3}$$

Similarly moves to upright position if $\lambda < \sqrt{3}$. If the toy is placed in a more upright position to start with, the same argument follows, but with a $\cos\theta$ on both sides.



We have $\tan\theta = \lambda$, $\varphi + \theta = \pi/2 \Rightarrow \tan\varphi = \frac{1}{\lambda}$.

So, the toy returns upright if

$$\left(\frac{3a}{8} - a\tan\varphi\right) \cdot \frac{2}{3}\pi a^3 \cdot \cos(\pi - \theta) > \left(\frac{\lambda a}{4} + a\tan\varphi\right) \cdot \frac{1}{3}\pi a^3 \lambda \cdot \cos(\pi - \theta)$$

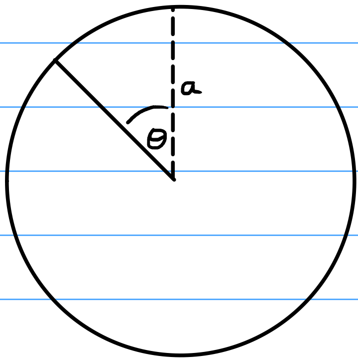
$$\iff \left(\frac{3}{8} - \frac{1}{\lambda}\right) \times 2 > \left(\frac{\lambda}{4} + \frac{1}{\lambda}\right) \times \lambda$$

$$\iff \frac{3}{4}\lambda - 2 > \frac{\lambda^3}{4} + \lambda$$

$$\iff \lambda^3 + \lambda + 8 < 0$$

Differentiating w.r.t λ, we obtain $3\lambda^2 + 1 > 0 \ \forall \lambda$, so the original function is increasing. As $f(0) = 8$, it is positive for all positive λ. So the toy never returns upright.

STEP II 1996 Q10



The vertical height above the centre of the wheel is $a\cos\theta$. The vertical component of velocity is $a\omega\sin\theta$

s	?
u	$a\omega\sin\theta$
v	0
a	$-g$
t	$\times$

$$v^2 = u^2 + 2as$$
$$0^2 = a^2\omega^2\sin^2\theta - 2gs$$
$$\Rightarrow s = \frac{a^2\omega^2\sin^2\theta}{2g}$$

So, the max height reached above the ground is $h = a\cos\theta + \frac{a^2\omega^2}{2g}\sin^2\theta$. Considering this as a fuction of θ,

$$\frac{dh}{d\theta} = -a\sin\theta + \frac{\omega^2 a^2}{g}\sin\theta\cos\theta = 0$$

So $\sin\cancel{\theta} = 0$ or $\cos\theta = \frac{g}{\omega^2 a}$ $\left(\text{For } \frac{g}{\omega^2 a} \leq 1\right)$

Hence $\sin^2\theta = 1 - \frac{g^2}{\omega^4 a^2}$

$$\text{So } h_{max} = a\cdot\frac{g}{\omega^2 a} + \frac{a^2\omega^2}{2g}\left(1 - \frac{g^2}{\omega^4 a^2}\right)$$
$$= \frac{g}{\omega^2} + \frac{a^2\omega^2}{2g} - \frac{g}{2\omega^2}$$
$$= \frac{g}{2\omega^2} + \frac{a^2\omega^2}{2g}.$$

To go higher than the top of the wheel, we require $h_{max} > a$, so

$$\frac{g}{2\omega^2} + \frac{\omega^2 a^2}{2g} > a$$

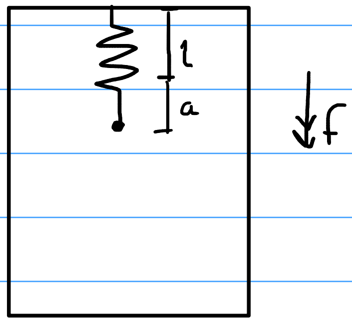
$$\Rightarrow a^2\omega^4 - 2ga\omega^2 - g^2 > 0$$
$$\Rightarrow (a\omega^2)^2 - 2g(a\omega^2) - g^2 > 0$$
$$\Rightarrow (a\omega^2 - g)^2 > 0$$

This is true for all $\omega \neq \sqrt{\frac{g}{a}}$. But we also know $\frac{g}{\omega^2 a} \leq 1 \Rightarrow \omega \geq \sqrt{\frac{g}{a}}$). So $\omega_0 = \sqrt{\frac{g}{a}}$.

The maximum possible height above the top of the wheel is

$$\frac{g}{2\omega^2} + \frac{a^2\omega^2}{2g} - a$$
$$= \frac{a}{2}\left[\frac{g}{a\omega^2} - 2 + \frac{\omega^2 a}{g}\right]$$
$$= \frac{a}{2}\left(\sqrt{\frac{a}{g}}\cdot\omega - \sqrt{\frac{g}{a}}\cdot\frac{1}{\omega}\right)^2$$
$$= \frac{a}{2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2, \text{ as required.}$$

STEP II 1996 Q11



We have $mg - T = m\left(\frac{d^2y}{dt^2} + f\right)$ $(*)$

In equilibrium $T = mg \Rightarrow mg = \frac{\lambda a}{l} \Rightarrow \frac{\lambda}{l} = \frac{mg}{a}$

So $T = \frac{\lambda y}{l} = \frac{mgy}{a}$.

So $(*)$ becomes $mg - \frac{mgy}{a} = m\left(\frac{d^2y}{dt^2} + f\right)$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{g}{a}y = g - f, \text{ as required.}$$

So $y = A\cos\sqrt{\frac{g}{a}}t + B\sin\sqrt{\frac{g}{a}}t + a - \frac{af}{g}$.

$y(0) = a,\ y'(0) = 0 \Rightarrow A = \frac{af}{g},\ B = 0$

so $y = \frac{af}{g}\cos\sqrt{\frac{g}{a}}t + a - \frac{af}{g}$, which has amplitude $\frac{af}{g}$.

The minimum value of y is thus $a\left(1 - \frac{2f}{g}\right)$. But $f < g/2$, so $y_{min} > 0\ \forall t$, so the string is never slack.

Now $f = 0$, so $y'' + \frac{g}{a}y = g \Rightarrow y = A\cos\sqrt{\frac{g}{a}}t + B\sin\sqrt{\frac{g}{a}}t + a$

$y(0) = \frac{af}{g}\cos\omega T + a - \frac{af}{g} \Rightarrow A = \frac{af}{g}\cos\omega T - \frac{af}{g}$

$\dot{y}(0) = -\frac{af}{g}\omega\sin\omega T \Rightarrow B\omega = -\frac{af}{g}\omega\sin\omega T$

$$\Rightarrow B = -\frac{af}{g}\sin\omega T.$$

So $y = \frac{af}{g}\left[(\cos\omega T - 1)\cos\omega t - \sin\omega T\sin\omega t\right] + a$

So amplitude is $\frac{af}{g}\sqrt{(\cos\omega T - 1)^2 + \sin^2\omega T}$

$$= \frac{af}{g}\sqrt{2 - 2\cos\omega T}$$

$$= \frac{af}{g}\sqrt{4\sin^2\tfrac{1}{2}\omega T}$$

$$= \frac{2af}{g}\left|\sin\tfrac{1}{2}\omega T\right|, \text{ as required}$$

STEP II 1996 Q12

(i) $X_1, \dots, X_n \overset{iid}{\sim} U(0,1)$
$Y = \max_i X_i$

$$P(Y<y) = P(X_1<y, X_2<y, \dots, X_n<y)$$
$$= P(X_1<y)P(X_2<y)\dots P(X_n<y) \quad \text{by independence}$$
$$= P(X_1<y)^n \quad \text{as all } U(0,1)$$
$$= y^n$$

So the density is ny^{n-1}.

$$Ey = \int_0^1 y \cdot ny^{n-1}\,dy = \left[\frac{ny^{n+1}}{n+1}\right]_0^1$$
$$= \frac{n}{n+1}$$
$$Ey^2 = \int_0^1 y^2 \cdot ny^{n-1}\,dy = \left[\frac{ny^{n+2}}{n+2}\right]_0^1$$
$$= \frac{n}{n+2}$$

$$\text{So } \operatorname{Var} Y = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2$$
$$= \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}$$
$$= \frac{n(n^2+2n+1-n^2-2n)}{(n+2)(n+1)^3}$$
$$= \frac{n}{(n+2)(n+1)^2}$$
, as required.

(ii) $P(X_i \le t) = 1 - e^{-t/\lambda}$
$$P(Y \le t) = 1 - P(X_1 \ge t, X_2 \ge t, \dots, X_n \ge t)$$
$$= 1 - (e^{-t/\lambda})^n$$
$$= 1 - e^{-tn/\lambda}$$
So the density of Y is $\frac{n}{\lambda}e^{-tn/\lambda}$.

$$EY = \int_0^\infty \frac{n}{\lambda} t e^{-nt/\lambda}\, dt \qquad u \quad t \qquad v' \quad \frac{n}{\lambda} e^{-nt/\lambda}$$

$$u' \quad 1 \qquad v \quad -e^{-nt/\lambda}$$

$$= \left[-te^{-nt/\lambda}\right]_0^\infty + \int_0^\infty e^{-nt/\lambda}\, dt$$

$$= 0 + \left[-\frac{\lambda}{n} e^{-nt/\lambda}\right]_0^\infty$$

$$= \lambda/n.$$

STEP II 1996 Q13

In $(1+t)^n(1+t)^n$, a term in t^n can be made from a t^k from the first bracket and a t^{n-k} from the second bracket. This has coefficient $\binom{n}{k}\binom{n}{n-k}$. Summing over all k, the coefficient is $\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\dots+\binom{n}{r}\binom{n}{n-r}+\dots+\binom{n}{n}\binom{n}{0}$. In $(1+t)^{2n}$, the coefficient of t^n is $\binom{2n}{n}$. As $(1+t)^n(1+t)^n=(1+t)^{2n}$, we have

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\dots+\binom{n}{r}\binom{n}{n-r}+\dots+\binom{n}{n}\binom{n}{0}=\binom{2n}{n}, \text{ as required.}$$

We can meet at any of the intersections on the line running from SW to NE diagonally. To reach the k^{th} one from the bottom, I must go north k times and west $n-k$ times. My friend must go east k times and south $n-k$ times. The probability of this happening is $\left(\frac{1}{2}\right)^n\binom{n}{k}\times\left(\frac{1}{2}\right)^n\binom{n}{n-k}$

$$=\left(\frac{1}{2}\right)^{2n}\binom{n}{k}\binom{n}{n-k}.$$

Hence the total probability of meeting is

$$\left(\frac{1}{2}\right)^{2n}\left[\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\dots+\binom{n}{r}\binom{n}{n-r}+\dots+\binom{n}{n}\binom{n}{0}\right]$$
$$=\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}.$$

Now, I leave $2k$ minutes later. Now the possible meeting places are still a SW–NE diagonal line, but shifted SE by k spaces. So now the probability of meeting is

$$\left(\frac{1}{2}\right)^{2n}\left[\binom{n-k}{0}\binom{n+k}{k}+\binom{n-k}{1}\binom{n+k}{k-1}+\dots+\binom{n-k}{n-k}\binom{n+k}{k}\right]$$

which is $\left(\frac{1}{2}\right)^{2n}\times$ the coefficient of t^n in $(1+t)^{n-k}(1+t)^{n+k}=(1+t)^{2n}$

So this is equal to $\left(\frac{1}{2}\right)^{2n}\times\binom{2n}{n}$, as before.

Now $n=1$ and I leave 1 minute late. After 1 minute my friend is 1 unit either west or north of me. At this point, both me and my friend must choose correctly to walk towards each other, each independently with probability $1/2$. So the probability that we meet is $1/4$.

STEP II 1996 Q14

$$EY^n = \frac{1}{4}\times\left(\frac{1}{4}\right)^n + \int_{1/4}^{3/4} y^n\,dy + \frac{1}{4}\times\left(\frac{3}{4}\right)^n$$

$$= \frac{1}{4^{n+1}} + \left[\frac{y^{n+1}}{n+1}\right]_{1/4}^{3/4} + \frac{3^n}{4^{n+1}}$$

$$= \frac{1}{4^{n+1}} + \frac{3^{n+1}}{(n+1)4^{n+1}} - \frac{1}{(n+1)4^{n+1}} + \frac{3^n}{4^{n+1}}$$

$$= \frac{1}{(n+1)4^{n+1}}\left(n+1+3^{n+1}-1+n\times 3^n+3^n\right)$$

$$= \frac{1}{(n+1)4^{n+1}}\left(n+3\times 3^n+3^n+n\times 3^n\right)$$

$$= \frac{n+3^n(4+n)}{(n+1)4^{n+1}}$$

so $EY = EY^1 = \dfrac{1+3\times 5}{2\times 16} = \dfrac{16}{32} = \dfrac{1}{2} = EX$

$$EY^2 = \frac{2+9\times 6}{3\times 64} = \frac{56}{192} = \frac{7}{24}$$

$$EX^2 = \int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \frac{1}{3}$$

so $EX^2 - EY^2 = \dfrac{1}{3} - \dfrac{7}{24} = \dfrac{8}{24} - \dfrac{7}{24} = \dfrac{1}{24}$.

$$EX^n - EY^n = \frac{1}{n+1} - \frac{n+(4+n)3^n}{(n+1)4^{n+1}}$$

$$= \frac{4^{n+1}-n-(4+n)3^n}{(n+1)4^{n+1}}$$

We want to show this is positive. The denominator is clearly positive, so consider the numerator.

So consider $4^{n+1} - n - (4+n)3^n$

$$= 4\times(3+1)^n - n - (4+n)3^n$$

$$= 4\left(3^n + n3^{n-1} + \binom{n}{2}3^{n-2} + \cdots\right) - n - (4+n)3^n$$

$$> 4\times 3^n + 4n3^{n-1} - n - 4\times 3^n - 3n\times 3^{n-1}$$

$$= n\times 3^{n-1} - n$$

$$= n(3^{n-1} - 1)$$

> 0 for $n \geq 2$. So $EX^n \geq EY^n$ for $n \geq 2$.

Now Y_i are iid $\sim Y$.

Then $Y_1 + \text{—} + Y_{24,000} \overset{d}{\approx} N(12{,}000, 24{,}000(\frac{7}{24} - \frac{1}{4}))$

$$= N(12{,}000, 1000)$$

$P(A < k) = 3/4$

$$\Rightarrow P\left(\frac{A - 12{,}000}{\sqrt{1000}} > \frac{k - 12{,}000}{\sqrt{1{,}000}}\right) = 3/4.$$

So $\dfrac{k - 12{,}000}{\sqrt{1{,}000}} = 0.6745$

$\Rightarrow k = 12{,}000 + 21.33$

so $k = 12{,}021.33$.