STEP I 1996 Q1

$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$

$S = 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi V^2}$

$= 2\pi r^2 + \frac{2V}{r}$

so $\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2} = 0$

$\Rightarrow 4\pi r^3 = 2V$

$\Rightarrow r = \left(\frac{V}{2\pi}\right)^{1/3}$

so $S = 2\pi\left(\frac{V}{2\pi}\right)^{2/3} + 2V\cdot\left(\frac{2\pi}{V}\right)^{1/3}$

$= (2\pi V^2)^{1/3} + 2(2\pi V^2)^{1/3}$

$= 3(2\pi V^2)^{1/3}$, as required.

we have $r = \left(\frac{V}{2\pi}\right)^{1/3}$, so $h = \frac{V}{\pi r^2} = \frac{V}{\pi} \times \left(\frac{2\pi}{V}\right)^{2/3}$

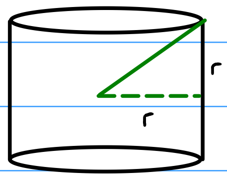
$= V^{1/3} \times \pi^{-1/3} \times 2^{2/3}$

$= V^{1/3} \times \pi^{-1/3} \times 2 \times 2^{-1/3}$

$= 2\left(\frac{V}{2\pi}\right)^{1/3}$

$= 2r$

So, the height of the cylinder is equal to the diameter. Thus a sphere of radius r just fits inside the tin, with volume $\frac{4}{3}\pi r^3$. The volume of the tin is $\pi r^2(2r) = 2\pi r^3$. $\frac{2}{3} \times 2\pi r^3 = \frac{4}{3}\pi r^3$, so the sphere has volume $\frac{2}{3}V$.



For the tin to fit inside the sphere, the radius of the sphere must be $\sqrt{2r^2} = \sqrt{2}r$, as shown. So the volume is $\frac{4}{3} \times \pi \times (\sqrt{2})^3 \times r^3 = 2\sqrt{2} \cdot \frac{4}{3}\pi r^3$ so the volume is $2\sqrt{2}$ times bigger than the previous sphere, so it has volume $\frac{4\sqrt{2}}{3}V$.

STEP I 1996 Q2

i) $\int_0^1 (1+(\alpha-1)x)^n dx = \left[\frac{1}{(n+1)(\alpha-1)}(1+(\alpha-1)x)^{n+1}\right]$

$= \frac{1}{(n+1)(\alpha-1)}(\alpha^{n+1}-1)$

ii)

$$\int_0^1 (\alpha x + (1-x))^n dx = \int_0^1 (1-x)^n + \binom{n}{1}(1-x)^{n-1} x\alpha + \dots + \binom{n}{k}(1-x)^{n-k} x^k \alpha^k + \dots + x^n \alpha^n dx$$

So the coefficient of α^n is $\int_0^1 \binom{n}{k}(1-x)^{n-k} x^k dx$

$= \binom{n}{k}\int_0^1 x^k (1-x)^{n-k} dx$

iii) Note that $\int_0^1 (\alpha x + (1-x))^n dx$

$= \int_0^1 (1+(\alpha-1)x)^n dx$

$= \frac{\alpha^{n+1}-1}{(n+1)(\alpha-1)}$ by (i).

But this is a polynomial of degree n, so

$$\alpha^{n+1} - 1 = (n+1)(\alpha-1)(a_0 + a_1\alpha + \text{———} + a_n\alpha^n)$$

$$= (n+1)(-a_0 + (a_0 - a_1)\alpha + \text{———} + (a_{n-1} - a_n)\alpha^{n-1} + a_n\alpha^n)$$

and so comparing coefficients, $a_0 = \frac{1}{n+1}$, $a_n = \frac{1}{n+1}$, and the rest of the a_i are also $\frac{1}{n+1}$.

Hence, $\binom{n}{k}\int_0^1 x^k (1-x)^{n-k} dx = \frac{1}{n+1}$

$\Rightarrow \int_0^1 x^k (1-x)^{n-k} dx = \frac{1}{n+1} \times \frac{k!(n-k)!}{n!}$

$= \frac{k!(n-k)!}{(n+1)!}$, as required.

STEP I 1996 Q3

Throughout this question, we use the notation $x|y$ to mean that x is a factor of y.

i) $n^5 - n^3 = n^3(n^2-1)$

$= n^3(n+1)(n-1) \quad = n^2 \times (n-1)n(n+1)$

We show that 3 and 8 are factors of n^5-n^3. Since 3 and 8 are coprime, this means that 24 is a factor of n^5-n^3.

Now, exactly one of $(n-1)$, n, and $(n+1)$ is a multiple of 3, so $3|(n-1)n(n+1) \Rightarrow 3|n^5-n^3$.

If n is even, then $8|n^3 \Rightarrow 8|n^5-n^3$, so $24|n^5-n^3$.

If n is odd, then one of $(n-1)$ and $(n+1)$ is a multiple of 4, and the other is a multiple of 2. Hence $8|(n-1)(n+1) \Rightarrow 24|n^5-n^3$.

ii) $2^{2n}-1 = (2^n+1)(2^n-1)$.

2^n only has 2 as a prime factor, so $3 \nmid 2^n$. Hence either 2^n+1 or 2^n-1 is a multiple of 3. So $3|2^{2n}-1$.

iii) $n^3-1 = (n-1)(n^2+n+1)$

By assumption, $3|n-1$, so $n = 3k+1$.

So $n^2+n+1 = (3k+1)^2+(3k+1)+1$

$= 9k^2+9k+3$

$= 3(3k^2+3k+1)$, so $3|(n^2+n+1)$.

$3|(n-1)$ and $3|(n^2+n+1)$, so $9|(n^3-1)$, as required.

STEP I 1996 Q4

For $|a|<1$, $\displaystyle\int_0^1 \frac{1}{x^2+2ax+1}\,dx = \int_0^1 \frac{1}{(x+a)^2+(1-a^2)}\,dx$ Set $u=x+a$, $du=dx$,

$$= \int_a^{1+a} \frac{1}{u^2+(1-a^2)}\,du$$

Set $u=\sqrt{1-a^2}\tan\theta$

$\Rightarrow \frac{du}{d\theta} = \sqrt{1-a^2}\sec^2\theta$

$$= \int_{\arctan\frac{a}{\sqrt{1-a^2}}}^{\arctan\frac{1+a}{\sqrt{1-a^2}}} \frac{\sqrt{1-a^2}\sec^2\theta}{(1-a^2)(\tan^2\theta+1)}\,d\theta$$

$$= \int_{\arctan\frac{a}{\sqrt{1-a^2}}}^{\arctan\frac{1+a}{\sqrt{1-a^2}}} \frac{d\theta}{\sqrt{1-a^2}}$$

$$= \frac{1}{\sqrt{1-a^2}}\left[\arctan\frac{1+a}{\sqrt{1-a^2}} - \arctan\frac{a}{\sqrt{1-a^2}}\right]$$

Now, $\tan(x-y) = \dfrac{\tan x - \tan y}{1+\tan x\tan y}$

Setting $x=\arctan p$, $y=\arctan q$, we have

$$\tan(\arctan p - \arctan q) = \frac{p-q}{1+pq}$$

so $\arctan p - \arctan q = \arctan\left(\dfrac{p-q}{1+pq}\right)$.

So $\dfrac{1}{\sqrt{1-a^2}}\left[\arctan\dfrac{1+a}{\sqrt{1-a^2}} - \arctan\dfrac{a}{\sqrt{1-a^2}}\right]$

$$= \frac{1}{\sqrt{1-a^2}}\arctan\left(\frac{\frac{1+a}{\sqrt{1-a^2}} - \frac{a}{\sqrt{1-a^2}}}{1+\frac{a(a+1)}{1-a^2}}\right)$$

$$= \frac{1}{\sqrt{1-a^2}}\arctan\left(\frac{\sqrt{1-a^2}}{1-a^2+a^2+a}\right)$$

$$= \frac{1}{\sqrt{1-a^2}} \arctan\left(\frac{\sqrt{1-a^2}}{\sqrt{(1+a)^2}}\right)$$

$$= \frac{1}{\sqrt{1-a^2}} \arctan\sqrt{\frac{(1+\cancel{a})(1-a)}{(1\cancel{+a})(1+a)}}$$

$$= \frac{1}{\sqrt{1-a^2}} \arctan\sqrt{\frac{1-a}{1+a}}, \quad \text{as required.}$$

For $|a|>1$, $I = \int_0^1 \frac{1}{(x+a)^2-(a^2-1)}\,dx$ Set $u=x+a$, $du=dx$

$$= \int_a^{1+a} \frac{1}{u^2-(a^2-1)}$$

Now $\frac{1}{u^2-(a^2-1)} \equiv \frac{A}{u+\sqrt{a^2-1}} + \frac{B}{u-\sqrt{a^2-1}}$

$$\Rightarrow 1 \equiv A(u-\sqrt{a^2-1}) + B(u+\sqrt{a^2-1})$$

$u=\sqrt{a^2-1} \Rightarrow B = \frac{1}{2\sqrt{a^2-1}}$

$u=-\sqrt{a^2-1} \Rightarrow A = \frac{-1}{2\sqrt{a^2-1}}$

So $I = \frac{1}{2\sqrt{a^2-1}} \int_a^{1+a} \frac{1}{u-\sqrt{a^2-1}} - \frac{1}{u+\sqrt{a^2-1}}\,du$

$$= \frac{1}{2\sqrt{a^2-1}} \Big[\ln|u-\sqrt{a^2-1}| - \ln|u+\sqrt{a^2-1}|\Big]_a^{1+a}$$

$$= \frac{1}{2\sqrt{a^2-1}} \Big[\ln|1+a-\sqrt{a^2-1}| - \ln|1+a+\sqrt{a^2-1}| - \ln|a-\sqrt{a^2-1}| + \ln|a+\sqrt{a^2-1}|\Big]$$

$$= \frac{1}{2\sqrt{a^2-1}} \ln\left|\frac{(1+a-\sqrt{a^2-1})(a+\sqrt{a^2-1})}{(1+a+\sqrt{a^2-1})(a-\sqrt{a^2-1})}\right|$$

$$= \frac{1}{2\sqrt{a^2-1}} \ln\left|\frac{a+\sqrt{a^2-1}+a^2+a\sqrt{a^2-1}-a\sqrt{a^2-1}-a^2+1}{a-\sqrt{a^2-1}+a^2-a\sqrt{a^2-1}+a\sqrt{a^2-1}-a^2+1}\right|$$

$$= \frac{1}{2\sqrt{a^2-1}} \ln\left|\frac{(a+1)+\sqrt{a^2-1}}{(a+1)-\sqrt{a^2-1}}\right|$$

$$= \frac{1}{2\sqrt{a^2-1}} \ln\left|\frac{(a+1)^2+2(a+1)\sqrt{a^2-1}+(a+1)(a-1)}{(a+1)^2-(a+1)(a-1)}\right| \quad \text{(rationalising the denominator)}$$

$$= \frac{1}{2\sqrt{a^2-1}} \ln \left| \frac{(a+1)(a+1+2\sqrt{a^2-1}+a-1)}{(a+1)(a+1-a+1)} \right|$$

$$= \frac{1}{2\sqrt{a^2-1}} \ln \left| \frac{2a+2\sqrt{a^2-1}}{2} \right|$$

$$= \frac{1}{2\sqrt{a^2-1}} \ln \left| a+\sqrt{a^2-1} \right|$$, as required.

STEP I 1996 Q5

i) $(r+s\sqrt{3})^2 = r^2 + 2rs\sqrt{3} + 3s^2$

$= r^2 + 3s^2 + 2rs\sqrt{3} = 4 - 2\sqrt{3}$

So, $r^2+3s^2 = 4$ and $2rs = -2 \Rightarrow r = -\frac{1}{s}$

$\Rightarrow \frac{1}{s^2} + 3s^2 = 4$

$\Rightarrow 3s^4 - 4s^2 + 1 = 0$

$\Rightarrow (3s^2-1)(s^2-1) = 0$

So $s^2 = 1/3$ or $s^2 = 1$, but $s^2 \neq 1/3$ as $s \in \mathbb{Q}$, so $s = \pm 1$.

If $s=1$, $r=-1$ and if $s=-1$, then $r=1$.

ii) $(p+qi)^2 = p^2 + 2pqi - q^2$

$= p^2 - q^2 + 2pqi = 3 - 2\sqrt{3} + (2-2\sqrt{3})i$

So, $p^2 - q^2 = 3 - 2\sqrt{3}$, $pq = 1 - \sqrt{3} \Rightarrow p = \frac{1-\sqrt{3}}{q}$.

So, $\frac{(1-\sqrt{3})^2}{q^2} - q^2 = 3 - 2\sqrt{3}$

$\Rightarrow q^4 + (3-2\sqrt{3})q^2 - (4-2\sqrt{3}) = 0$

Note that $q^2 = 1$ is a solution, so factorises to $(q^2-1)(q^2-\alpha)$

$\Rightarrow q^4 - (\alpha+1)q^2 + \alpha = q^4 + (3-2\sqrt{3})q^2 - (4-2\sqrt{3})$

$\Rightarrow \alpha = -4 + 2\sqrt{3}$

So $q^2 = 1$ or $q^2 = -4 + 2\sqrt{3}$, but $-4 + 2\sqrt{3} < 0$ and $q \in \mathbb{R}$, so $q = \pm 1$.

If $q = 1$, $p = 1 - \sqrt{3}$

If $q = -1$, $p = \sqrt{3} - 1$

iii) $(1+i)z^2 - 2z + 2\sqrt{3} - 2 = 0$

$\Rightarrow z = \frac{2 \pm \sqrt{4 - 4(1+i)(2\sqrt{3}-2)}}{2(1+i)}$

$$= \frac{2 \pm 2\sqrt{1-2\sqrt{3}+2-2\sqrt{3}i+2i}}{2(1+i)}$$

$$= \frac{1 \pm \sqrt{(3-2\sqrt{3})+2(1-\sqrt{3})i}}{1+i}$$

$$= \frac{1 \pm (1-\sqrt{3}+i)}{2}$$ (by using the previous result)

$$= \frac{2-\sqrt{3}+i}{1+i} \quad \text{or} \quad \frac{\sqrt{3}-i}{1+i}$$

$$= \frac{(2-\sqrt{3}+i)(1-i)}{(1+i)(1-i)} \quad \text{or} \quad \frac{(\sqrt{3}-i)(1-i)}{(1+i)(1-i)}$$

$$= \frac{2-2i-\sqrt{3}+i\sqrt{3}+i+1}{2} \quad \text{or} \quad \frac{\sqrt{3}-i\sqrt{3}-i-1}{2}$$

$$= \frac{(3-\sqrt{3})+(-1+\sqrt{3})i}{2} \quad \text{or} \quad \frac{(-1+\sqrt{3})-(1+\sqrt{3})i}{2}$$

STEP I 1996 Q6

i) $\sin\frac{x}{2}\left(1+2\sum_{k=1}^{n}\cos kx\right)$

$= \sin\frac{x}{2} + 2\sum_{k=1}^{n}\sin\frac{x}{2}\cos kx$

$= \sin\frac{x}{2} + \sum_{k=1}^{n}\sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x$

$= \sin\frac{x}{2} + \sin(n+\frac{1}{2})x - \sin\frac{1}{2}x$

$= \sin(n+\frac{1}{2})x$

So $\sin(\frac{x}{2})\left(1+2\sum_{k=1}^{n}\cos kx\right) = \sin(n+\frac{1}{2})x \Rightarrow \frac{\sin(n+\frac{1}{2})x}{\sin(x/2)} = 1+2\sum_{k=1}^{n}\cos kx$, as required.

ii) $\int_0^{\pi} f(x)\,dx = \int_0^{\pi} 1\,dx + 2\sum_{k=1}^{n}\int_0^{\pi}\cos kx\,dx$

$= \pi + 2\sum_{k=1}^{n}\left[\frac{1}{k}\sin kx\right]_0^{\pi}$

$= \pi.$

$\int_0^{\pi} f(x)\cos x\,dx = \int_0^{\pi}\cos x\,dx + 2\sum_{k=1}^{n}\int_0^{\pi}\cos x\cos kx\,dx$

$= 0 + \sum_{k=1}^{n}\int_0^{\pi}\cos(k+1)x + \cos(k-1)x\,dx$

$= \int_0^{\pi}\cos 2x + 1 + \sum_{k=2}^{n}\left[\frac{1}{k+1}\sin(k+1)x + \frac{1}{(k-1)}\sin(k-1)x\right]_0^{\pi}$

$= 0 + \pi + 0$

$= \pi.$

STEP I 1996 Q7

i) $\frac{dy}{dt} = -ky$

$\Rightarrow \frac{1}{y} dV = -k dt$

$\Rightarrow \ln y = -kt + c$

$\Rightarrow y = Ae^{-kt}$

$y(0) = 1 \Rightarrow A = 1$

so $y = e^{-kt} = (e^{-k})^t$ so $b = e^{-k}$ (and $0 < e^{-k} < 1$ as $k > 0$)

ii) $\frac{dy}{dt} = -ky + a$

$\Rightarrow \frac{dy}{dt} = y \ln b + a$

$\Rightarrow \frac{dy}{dt} - y \ln b = a$ $\qquad$ $IF = e^{\int -\ln b \, dt} = e^{-t\ln b}$

so $\frac{d}{dt}(ye^{-t\ln b}) = ae^{-t\ln b}$

$\Rightarrow ye^{-t\ln b} = \frac{-1}{\ln b} ae^{-t\ln b} + C$

$\Rightarrow y = \frac{-a}{\ln b} + Ce^{t\ln b}$

$y(0) = 1 \Rightarrow C = 1 + \frac{a}{\ln b}$

so $y = \frac{-a}{\ln b} + \left(1 + \frac{a}{\ln b}\right)e^{t\ln b}$

$= \frac{-a}{\ln b} + \left(1 + \frac{a}{\ln b}\right) \cdot b^t$

STEP I 1996 Q8

i) $0.383838\ldots = \sum_{n=0}^{\infty} 0.38 \times \left(\frac{1}{100}\right)^n$

$$= \frac{0.38}{1-\frac{1}{100}} = \frac{38}{99}.$$

ii) $x = 0.a_1a_2a_3\ldots\underbrace{a_N a_{N+1} a_{N+2} \ldots\ldots a_{N+k-1}}_{\text{repeating unit}}\ldots\ldots$

$$= \frac{a_1a_2\ldots a_N}{10^N} + \frac{1}{10^N} \times 0.a_N \ldots a_{N+k-1}\ldots\ldots$$

$$= \frac{a_1a_2\ldots a_N}{10^N} + \frac{1}{10^N}\sum_{n=0}^{\infty}(0.a_N\ldots a_{N+k-1}) \cdot \left(\frac{1}{10^k}\right)^n$$

$$= \frac{a_1a_2\ldots a_N}{10^N} + \frac{1}{10^N} \cdot \frac{a_N \ldots a_{N+k-1}}{10^k} \cdot \frac{1}{1-\frac{1}{10^k}}$$

$$= \frac{a_1a_2\ldots a_N}{10^N} + \frac{a_N\ldots a_{N+k-1}}{10^N(10^k-1)}$$

Both of these terms are rational, so their sum is rational.

STEP I 1996 Q9

By conservation of energy,

$$\frac{1}{2}mv^2 = mgl$$
$$\Rightarrow v = \sqrt{2gl}$$

Now, the elastic potential energy in a rope is $\frac{\lambda x^2}{2l} = \frac{mgx^2}{2kl}$.

So, $\frac{mg(h-l)^2}{2kl} = mgh$

$$\Rightarrow (h-l)^2 = 2khl$$
$$\Rightarrow h^2 - 2hl(1+k) + l^2 = 0$$
$$\Rightarrow (h - l(1+k))^2 - l^2(1+k)^2 + l^2 = 0$$
$$\Rightarrow (h - l(1+k))^2 = l^2(k^2+2k)$$

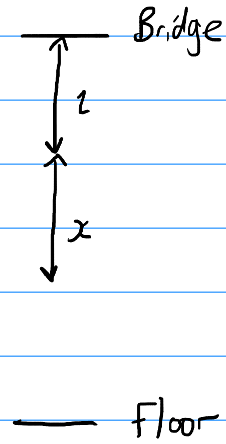
so $h = l(1+k) \pm l\sqrt{k^2+2k}$

$= l(1 + k \pm \sqrt{k^2+2k})$

but $h > l$, so

$h = l(1 + k + \sqrt{k^2+2k})$

By Newton's second law, $mg - T = ma$. But at the point of maximum speed, $a = 0$, so $T = mg$



$$\Rightarrow mg = \frac{mgx}{kl}$$
$$\Rightarrow x = kl$$

so distance from bridge is $kl + l = l(1+k)$.

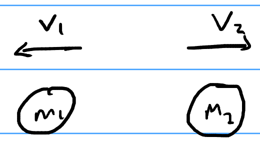
Then, considering energy,

$$\frac{1}{2}mv^2 + \frac{mg(kl)^2}{2kl} = mgl(1+k)$$

$$\Rightarrow v^2 = 2gl + 2kgl - gkl$$
$$= (k+2)gl.$$

STEP I 1996 Q10

By conservation of momentum, both parts are moving in opposite directions.



By conservation of momentum,

$$m_1 v_1 = m_2 v_2 = (M - m_1) v_2$$

$$\Rightarrow v_2 = \frac{m_1}{M - m_1} v_1$$

By conservation of energy,

$$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} \cdot (M - m_1) \cdot \left(\frac{m_1}{M - m_1}\right)^2 v_1^2$$

$$\Rightarrow 2E = m_1 v_1^2 \left(1 + \frac{m_1}{M - m_1}\right)$$

$$= m_1 v_1^2 \left(\frac{M}{M - m_1}\right).$$

Now $v_1 + v_2 = v_1\left(1 + \frac{m_1}{M - m_1}\right)$

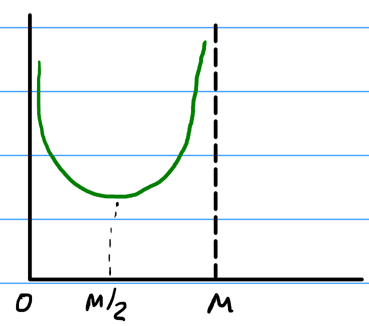
$$= \frac{M}{M - m_1} v_1$$

$$= \frac{2E}{m_1 v_1}$$

$$= \frac{2E}{m_1} \cdot \frac{M}{M - m_1} \cdot \frac{M - m_1}{M} \cdot \frac{1}{v_1}$$

$$= \frac{2EM}{m_1 (M - m_1)} \cdot \frac{1}{v_1 + v_2}$$

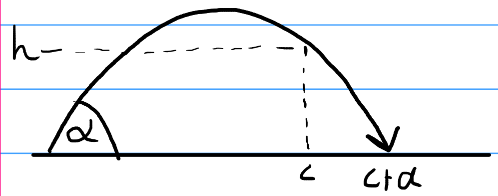
$$\Rightarrow (v_1 + v_2)^2 = \frac{2EM}{m_1 (M - m_1)}$$



The minimum occurs at $m_1 = M/2$, so

$$v_1 + v_2 = \sqrt{\frac{2EM}{\frac{M}{2} \cdot \frac{M}{2}}} = \sqrt{\frac{8E}{M}}$$, as required.

STEP I 1996 Q11



First considering when the particle hits the ground.

Horizontally, $s = ut \Rightarrow c+d = v\cos\alpha t$

$\Rightarrow t = \frac{c+d}{v\cos\alpha}$

Vertically, $s = ut + \frac{1}{2}at^2 \Rightarrow 0 = v\sin\alpha t - \frac{1}{2}gt^2$

$\Rightarrow 0 = (c+d)\tan\alpha - \frac{(c+d)^2 g}{2v^2}\sec^2\alpha \quad (1)$

Now considering when the particle passes through (c,h),

Horizontally, $s = ut \Rightarrow c = v\cos\alpha t$

$\Rightarrow t = \frac{c}{v\cos\alpha}$

Vertically $s = ut + \frac{1}{2}at^2 \Rightarrow h = v\sin\alpha t - \frac{1}{2}gt^2$

$\Rightarrow h = c\tan\alpha - \frac{c^2 g}{2v^2}\sec^2\alpha \quad (2)$

$(1) \Rightarrow \tan\alpha = \frac{(c+d)g}{2v^2}\sec^2\alpha$

$\Rightarrow \frac{g}{2v^2} = \frac{\tan\alpha}{(c+d)\sec^2\alpha}$

substituting into (2),

$$h = c\tan\alpha - \frac{c^2\sec^2\alpha\tan\alpha}{(c+d)\sec^2\alpha}$$

$$= c\tan\alpha\left(1 - \frac{c}{c+d}\right)$$

$$= \frac{cd\tan\alpha}{c+d}$$

$\Rightarrow \tan\alpha = \frac{h(c+d)}{cd}$, as required.

$$v^2 = \frac{g}{2}\left(\frac{(c+d)\sec^2\alpha}{\tan\alpha}\right)$$

$$= \frac{g}{2}\left(\frac{(c+d)(1+\tan^2\alpha)}{\tan\alpha}\right)$$

$$= \frac{g}{2} \frac{(c+d)\left(1+\frac{h^2(c+d)^2}{c^2d^2}\right)}{h(c+d)/cd}$$

$$= \frac{g}{2} \cdot \frac{cd\left(1+\frac{h^2(c+d)^2}{c^2d^2}\right)}{h}$$

$$= \frac{g}{2}\left(\frac{cd}{h} + \frac{h(c+d)^2}{cd}\right)$$, as required.

STEP I 1996 Q12

There are $\binom{m}{n}$ possible mark assignments. If the highest mark is k, there are $\binom{k-1}{n-1}$ ways of assigning marks to the remaining $n-1$ scripts. As all outcomes are equally likely, we have

$$P(K=k) = \binom{k-1}{n-1} \Big/ \binom{m}{n}$$

We know that $\sum_{k=n}^{m} P(K=k) = 1$

$$\Rightarrow \frac{1}{\binom{m}{n}} \sum_{k=n}^{m} \binom{k-1}{n-1} = 1$$

$$\Rightarrow \sum_{k=n}^{m} \binom{k-1}{n-1} = \binom{m}{n}.$$

$$EK = \sum_{k=n}^{m} kP(K=k)$$

$$= \frac{1}{\binom{m}{n}} \sum_{k=n}^{m} k\binom{k-1}{n-1}$$

$$= \frac{1}{\binom{m}{n}} \sum_{k=n}^{m} \frac{k!}{(n-1)!(k-n)!}$$

$$= \frac{1}{\binom{m}{n}} \sum_{k=n}^{m} n \cdot \frac{k!}{n!(k-n)!}$$

$$= \frac{n}{\binom{m}{n}} \sum_{k=n}^{m} \binom{k}{n}$$

$$= \frac{n}{\binom{m}{n}} \sum_{k=n+1}^{m+1} \binom{k+1}{(n+1)-1}$$

$$= \frac{n}{\binom{m}{n}} \cdot \binom{m+1}{n+1} \qquad \text{(by the previous result)}$$

$$= n \cdot \frac{n!(m-n)!}{m!} \cdot \frac{(m+1)!}{(n+1)!(m-n)!} = \frac{n}{n+1} \cdot (m+1).$$

STEP I 1996 Q13

$P(N=2) = \frac{1}{2}$

$P(N=3) = \frac{1}{2} \times \frac{2}{3}$

$P(N=4) = \frac{1}{2} \times \frac{1}{3} \times \frac{3}{4}$

so $P(N=k) = \frac{(k-1)}{k!}$.

$$EN = \sum_{k=2}^{\infty} k P(N=k)$$

$$= \sum_{k=2}^{\infty} \frac{1}{(k-2)!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}$$

Note that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

So $e^1 = \sum_{k=0}^{\infty} \frac{1}{k!}$

$= e.$

$$EN^2 = \sum_{k=2}^{\infty} \frac{k}{(k-2)!} = \frac{2}{0!} + \sum_{k=3}^{\infty} \frac{k}{(k-2)!}$$

$$= 2 + \sum_{k=3}^{\infty} \frac{k-2+2}{(k-2)!}$$

$$= 2 + \sum_{k=3}^{\infty} \frac{1}{(k-3)!} + \sum_{k=3}^{\infty} \frac{2}{(k-2)!}$$

$$= 2 + \sum_{k=1}^{\infty} \frac{1}{k!} + 2\left(\sum_{k=0}^{\infty} \frac{1}{k!} - 1\right)$$

$$= 2 + e + 2(e-1)$$

$$= 3e$$

So $\mathrm{Var}\,N = EN^2 - (EN)^2$

$= 3e - e^2$

$= e(3-e)$

STEP I 1996 Q14

$P(A|H) = \underset{\substack{\text{throw } r-1 \text{ more} \\ \text{heads}}}{p^{r-1}} + \underset{\substack{\text{throw a tail before} \\ r-1 \text{ heads}}}{(1-p^{r-1})} P(A|T)$

$P(A|T) = \underset{\substack{\text{throw a head before} \\ s-1 \text{ tails}}}{(1-q^{s-1})} P(A|H)$

So $P(A|H) = p^{r-1} + (1-p^{r-1})(1-q^{s-1})P(A|H)$

$\Rightarrow P(A|H) = \dfrac{p^{r-1}}{1-(1-p^{r-1})(1-q^{s-1})}$

$\Rightarrow P(A|T) = (1-q^{s-1})P(A|H)$

$= \dfrac{p^{r-1}(1-q^{s-1})}{1-(1-p^{r-1})(1-q^{s-1})}$

$P(A) = pP(A|H) + qP(A|T)$

$= \dfrac{p^r + qp^{r-1}(1-q^{s-1})}{1-(1-p^{r-1})(1-q^{s-1})}$.