STEP III 1995 Q1

$a^2x + ay + z = a^2$ ①
$ax + y + bz = 1$ ②
$a^2bx + y + bz = b$ ③

③ − ② $\Rightarrow x(a^2b - a) = b - 1$
$\Rightarrow a(ab-1)x = b-1$ (†)
$\Rightarrow x = \frac{b-1}{a(ab-1)}$ As long as $a \neq 0$, $ab \neq 1$

① − a×② $\Rightarrow z - abz = a^2 - a$
$\Rightarrow z(1-ab) = a(1-a)$
$\Rightarrow z = \frac{a(1-a)}{(1-ab)}$ As long as $ab \neq 1$

③ − b×① $\Rightarrow y - aby = b - a^2b$
$y(1-ab) = b(1-a^2)$
$\Rightarrow y = \frac{b(1-a^2)}{1-ab}$ As long as $ab \neq 1$

So, for $a \neq 0$, $ab \neq 1$, $x = \frac{b-1}{a(ab-1)}$, $y = \frac{b(1-a^2)}{1-ab}$, $z = \frac{a(1-a)}{1-ab}$

If $a=0$, (†) becomes $0 = b-1 \Rightarrow b=1$. So we have $x \in \mathbb{R}$, $y=1$, $z=0$. this is the line $\underline{r} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$.

If $ab=1$, (†) becomes $0 = b-1 \Rightarrow b=1 \Rightarrow a=1$. Then all three equations become $x+y+z=1$, which is a plane.

STEP III 1995 Q2

$I_0 = \int_0^a x^{1/2}(a-x)^{1/2}\,dx$. Set $x = a\sin^2 u$, $\frac{dx}{du} = 2a\sin u\cos u$

$= \int_0^{\pi/2} a^{1/2}\sin u(a-a\sin^2 u)^{1/2}\cdot 2a\sin u\cos u\,du$

$= 2a^2\int_0^{\pi/2}\sin^2 u\cos^2 u\,du$

$= \frac{a^2}{2}\int_0^{\pi/2}\sin^2 2u\,du$

$= \frac{a^2}{4}\int_0^{\pi/2} 1-\cos 4u\,du$

$= \frac{a^2}{4}\left[u-\frac{1}{4}\sin 4u\right]_0^{\pi/2}$

$= \frac{a^2}{4}\cdot\frac{\pi}{2}$

$= \frac{\pi a^2}{8}$, as required.

$I_n = \int_0^a x^{n+\frac{1}{2}}(a-x)^{1/2}\,dx$

$u = x^{n+\frac{1}{2}}$, $v' = (a-x)^{1/2}$

$u' = (n+\frac{1}{2})x^{n-1+\frac{1}{2}}$, $v = -\frac{2}{3}(a-x)^{3/2}$

$= -\frac{2}{3}\left[\underbrace{x^{n+\frac{1}{2}}(a-x)^{3/2}}_{=0}\right]_0^a + \frac{2}{3}(n+\frac{1}{2})\int_0^a x^{n-1+\frac{1}{2}}(a-x)^{3/2}\,dx$

$= \frac{(2n+1)}{3}\int_0^a (a-x)x^{n-1+\frac{1}{2}}(a-x)^{1/2}\,dx$

$= \frac{2n+1}{3}\left[a\int_0^a x^{n-1+\frac{1}{2}}(a-x)^{1/2}\,dx - \int_0^a x^{n+\frac{1}{2}}(a-x)^{1/2}\,dx\right]$

$= \frac{2n+1}{3}(aI_{n-1} - I_n)$

So $3I_n = (2n+1)aI_{n-1} - (2n+1)I_n$

$\Rightarrow (2n+4)I_n = (2n+1)aI_{n-1}$.

$I_0 = \frac{\pi a^2}{8}$

$I_1 = \frac{3}{6} \cdot \frac{\pi a^3}{8}$

$I_2 = \frac{5}{8} \times \frac{3}{6} \times \frac{\pi a^4}{8}$

$I_3 = \frac{7}{10} \times \frac{5}{8} \times \frac{3}{6} \times \frac{\pi a^5}{8}$

So $I_n = \frac{3 \times 5 \times \ldots \times (2n+1)}{6 \times 8 \times \ldots \times (2n+4)} \cdot a^{n+2} \cdot \frac{\pi}{8}$.

STEP III 1995 Q3

The auxiliary equation is $m^2+2km+1=0$

$$\Rightarrow m = \frac{-2k \pm \sqrt{4k^2-4}}{2}$$
$$= -k \pm \sqrt{k^2-1}$$

For $k>1$, these are distinct real roots, so $x = Ae^{(-k+\sqrt{k^2-1})t} + Be^{(-k-\sqrt{k^2-1})t}$

For $k=1$, this is a repeated real root, so $x=(A+Bt)e^{-kt} = (A+Bt)e^{-t}$

For $0<k<1$, the roots are complex, so $x = e^{-kt}(A\cos\sqrt{1-k^2}t + B\sin\sqrt{1-k^2}t)$

Now $x(0)=0 \Rightarrow A=0$, so $x = Be^{-kt}\sin\sqrt{1-k^2}t$.

Then $\frac{dx}{dt} = -kBe^{-kt}\sin\sqrt{1-k^2}t + \sqrt{1-k^2}Be^{-kt}\cos\sqrt{1-k^2}t = 0$

$$\Rightarrow -k\sin\sqrt{1-k^2}t + \sqrt{1-k^2}\cos\sqrt{1-k^2}t = 0$$
$$\Rightarrow \tan\sqrt{1-k^2}t = \frac{\sqrt{1-k^2}}{k}$$
so $\sqrt{1-k^2}t = \arctan\frac{\sqrt{1-k^2}}{k} + (n-1)\pi$
$$\Rightarrow t = \frac{1}{\sqrt{1-k^2}}\left[\arctan\frac{\sqrt{1-k^2}}{k} + (n-1)\pi\right]$$

So, $$\left|\frac{x_{n+1}}{x_n}\right| = \frac{B\exp\frac{-k}{\sqrt{1-k^2}}\left(\arctan\frac{\sqrt{1-k^2}}{k} + n\pi\right)}{B\exp\frac{-k}{\sqrt{1-k^2}}\left(\arctan\frac{\sqrt{1-k^2}}{k} + (n-1)\pi\right)} \cdot \left|\frac{\sin\left[\arctan\frac{\sqrt{1-k^2}}{k} + n\pi\right]}{\sin\left[\arctan\frac{\sqrt{1-k^2}}{k} + (n-1)\pi\right]}\right|$$

Note that $\sin(t+\pi) = -\sin t$, so with the modulus brackets the second fraction is 1.

So $\alpha = \exp\left(\frac{-k\pi}{\sqrt{1-k^2}}\right) \Rightarrow \ln(\alpha) = \frac{-k\pi}{\sqrt{1-k^2}}$
$$\Rightarrow \ln(\alpha)^2 = \frac{k^2\pi^2}{1-k^2}$$
$$\Rightarrow \ln(\alpha)^2 - k^2\ln(\alpha)^2 = k^2\pi^2$$
$$\Rightarrow \ln(\alpha)^2 = k^2(\pi^2 + \ln(\alpha)^2)$$

$$\Rightarrow k^2 = \frac{\ln(\alpha)^2}{\pi^2 + \ln(\alpha)^2}$$, as required.

STEP III 1995 Q4

$C_n(\theta) = \sum_{k=0}^{n} \cos k\theta$

so $C_n(\theta)\sin(\frac{1}{2}\theta) = \cos(0\theta)\sin\frac{1}{2}\theta + \cos\theta\sin\frac{1}{2}\theta + \cos 2\theta \sin\frac{1}{2}\theta + \dots + \cos n\theta \sin\frac{1}{2}\theta$

$= \frac{1}{2}\left((\sin\frac{1}{2}\theta + \sin\frac{1}{2}\theta) + (\sin\frac{3}{2}\theta - \sin\frac{1}{2}\theta) + (\sin\frac{5}{2}\theta - \sin\frac{3}{2}\theta) + \dots + \sin(n+\frac{1}{2})\theta - \sin(n-\frac{1}{2})\theta\right)$

(because $\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$)

$= \frac{1}{2}(\sin\frac{1}{2}\theta + \sin(n+\frac{1}{2})\theta)$

$= \sin\frac{1}{2}(n+1)\theta \cos\frac{n}{2}\theta$

so $C_n(\theta) = \dfrac{\sin\frac{1}{2}(n+1)\theta\cos\frac{n}{2}\theta}{\sin\frac{1}{2}\theta}$, as required.

$S_n(\theta)\sin(\frac{1}{2}\theta) = \sin(0\theta)\sin\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta + \sin 2\theta \sin\frac{1}{2}\theta + \dots + \sin n\theta \sin\frac{1}{2}\theta$

$= \frac{1}{2}\left(0 + (\cos\frac{1}{2}\theta - \cos\frac{3}{2}\theta) + (\cos\frac{3}{2}\theta - \cos\frac{5}{2}\theta) + \dots + (\cos(n+\frac{1}{2}\theta) - \cos(n-\frac{1}{2})\theta\right)$

$= \frac{1}{2}(\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta)$

$= -\sin\frac{1}{2}(n+1)\theta \sin -\frac{1}{2}n\theta$

$= \sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta$

So $S_n(\theta) = \dfrac{\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$

Now $|C_n(\theta) - \frac{1}{2}| = \left|\dfrac{\cos\frac{1}{2}n\theta\sin\frac{1}{2}(n+1)\theta}{\sin\frac{1}{2}\theta} - \frac{1}{2}\right|$

$= \left|\dfrac{2\cos\frac{1}{2}n\theta\sin\frac{1}{2}(n+1)\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}\right|$

$= \left|\dfrac{\sin(n+\frac{1}{2})\theta + \sin\frac{1}{2}\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}\right|$

$= \dfrac{|\sin(n+\frac{1}{2})\theta|}{2\sin\frac{1}{2}\theta} \leq \dfrac{1}{2\sin\frac{1}{2}\theta}$ (as $0 < \theta < 2\pi$, the denominator is positive).

STEP III 1995 Q5

$y = \sin^2(m\arcsin x)$

$\frac{dy}{dx} = 2\sin(m\arcsin x)\cos(m\arcsin x)\cdot\frac{m}{\sqrt{1-x^2}}$

$\frac{d^2y}{dx^2} = 2\cos^2(m\arcsin x)\cdot\frac{m^2}{1-x^2} - 2\sin^2(m\arcsin x)\cdot\frac{m^2}{1-x^2} + 2\sin(m\arcsin x)\cos(m\arcsin x)\cdot\frac{mx}{(1-x^2)^{3/2}}$

So $(1-x^2)y^{(2)} = 2m^2\left(\cos^2(m\arcsin x) - \sin^2(m\arcsin x)\right) + \frac{1}{\sqrt{1-x^2}}\cdot 2mx\sin(m\arcsin x)\cos(m\arcsin x)$

And $xy^{(1)} + 2m^2(1-2y)$

$= \frac{1}{\sqrt{1-x^2}}\cdot 2mx\sin(m\arcsin x)\cos(m\arcsin x) + 2m^2(1-2\sin^2(m\arcsin x))$

$= \frac{1}{\sqrt{1-x^2}}\cdot 2mx\sin(m\arcsin x)\cos(m\arcsin x) + 2m^2\left(\cos^2(m\arcsin x) - \sin^2(m\arcsin x)\right)$

So $(1-x^2)y^{(2)} = xy^{(1)} + 2m^2(1-2y)$, as required.

We now proceed via induction.

Differentiating the above result,

$$(1-x^2)y^{(3)} - 2xy^{(2)} = y^{(1)} + xy^{(2)} - 4m^2y^{(1)}$$
$$\Rightarrow (1-x^2)y^{(3)} = 3xy^{(2)} + (1-4m^2)y^{(1)}$$, so true for $n=1$.

We assume true for $n=k$, so

$$(1-x^2)y^{(k+2)} = (2k+1)xy^{(k+1)} + (k^2-4m^2)y^{(k)}.$$

Differentiating,

$$(1-x^2)y^{(k+3)} - 2xy^{(k+2)} = (2k+1)y^{(k+1)} + (2k+1)xy^{(k+2)} + (k^2-4m^2)y^{(k+1)}$$

$$\Rightarrow (1-x^2)y^{(k+3)} = (2k+3)y^{(k+2)} + (2k+1+k^2-4m^2)y^{(k+1)}$$

$$\Rightarrow (1-x^2)y^{(k+3)} = (2(k+1)+1)y^{(k+2)} + ((k+1)^2-4m^2)y^{(k+1)}$$, so true for $n=k+1$.

True for $n=1$, and if true for $n=k$ then true for $n=k+1$, so true $\forall n \in \mathbb{N}$ by induction.

Now $y(0)=0$, $y^{(1)}=0$, $y^{(2)}=2m^2$.

Further, at $x=0$ the relation simplifies to

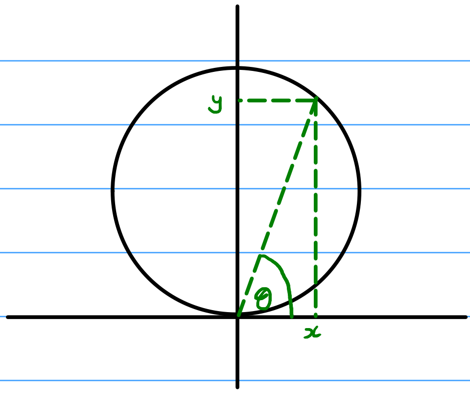
$$y^{(n+2)} = (n^2-4m^2)\,y^{(n)}$$

So all odd powers of $x$ have zero coefficient, so

$$y = 2m^2\frac{x^2}{2!} + \frac{(2^2-4m^2)2m^2}{4!} + \frac{(4^2-4m^2)(2^2-2m^2)}{6!}x^6 + \dots$$

$$= \sum_{r=1}^{\infty} 2m^2\cdot\left(\prod_{i=1}^{r-1}\left((2i)^2-4m^2\right)\right)\cdot\frac{x^{2r}}{(2r)!}$$

# STEP III 1995 Q6




We have $x^2+(y-1)^2=1$

$\Rightarrow x^2+y^2-2y+1=1$

$\Rightarrow x^2+y^2=2y$

$\Rightarrow r^2=2y$

but $\tan\theta=\frac{y}{x} \Rightarrow \tan^2\theta=\frac{y^2}{x^2}$

$=\frac{y^2}{1-(y-1)^2}$

$=\frac{y^2}{2y-y^2}$

$=\frac{y}{2-y}$

So $2\tan^2\theta-y\tan^2\theta=y$

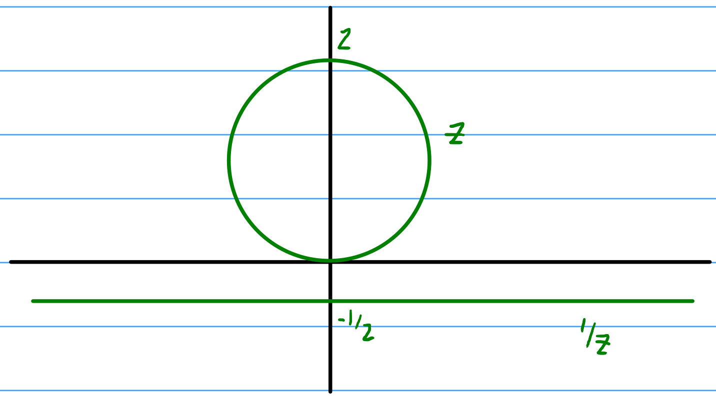
$\Rightarrow y=\frac{2\tan^2\theta}{1+\tan^2\theta}=\frac{2\tan^2\theta}{\sec^2\theta}=2\sin^2\theta$

So $r^2=4\sin^2\theta$

$\Rightarrow r=2\sin\theta$ (as $r>0$).

The modulus of $\frac{1}{z}$ is $\frac{1}{|z|}=\frac{1}{2\sin\theta}$, and the argument is $-\theta$.

Note $y=r\sin\theta$, so for $\frac{1}{z}$, $y=\frac{1}{2\sin\theta}\cdot\sin(-\theta)=\frac{-1}{2}$. So $\frac{1}{z}$ is the line $y=-\frac{1}{2}$.




Now we have $\frac{1}{w} = -1 + it$

$$\Rightarrow \quad w = \frac{1}{-1+it}$$

$$= \frac{1}{-1+it} \times \frac{-1-it}{-1-it}$$

$$= \frac{-1-it}{1+t^2}$$

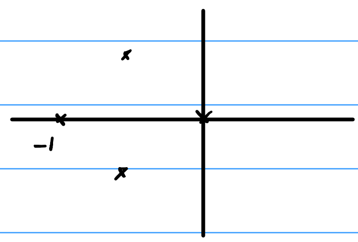
So the locus is $\left(\frac{-1}{1+t^2}, \frac{-t}{1+t^2}\right)$ for $t \in \mathbb{R}$.

For $t=0$ : $(-1, 0)$

$t=1$ : $(-\tfrac{1}{2}, -\tfrac{1}{2})$

$t=-1$ : $(-\tfrac{1}{2}, \tfrac{1}{2})$

$t \to \infty$ : $(0,0)$




So consider a circle, radius $\frac{1}{2}$, centre $(-\tfrac{1}{2}, 0)$. Then we need $(x+\tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}$.

$(x+\tfrac{1}{2})^2 + y^2$

$$= \frac{1}{(1+t^2)^2}\left[\left(\tfrac{1}{2}t^2 - \tfrac{1}{2}\right)^2 + t^2\right]$$

$$= \frac{1}{4(1+t^2)^2} \cdot \left[t^4 - 2t^2 + 1 + 4t^2\right]$$

$$= \frac{1}{4(1+t^2)^2}(1+t^2)^2$$

$= 1/4$, as required.

So the locus of $w$ is a circle with radius $\frac{1}{2}$ centred at $(-\tfrac{1}{2}, 0)$.

STEP III 1995 Q7

For each set, associativity follows from associativity of multiplication, so we will consider the identity, closure, and inverses.

i) This is a group. The identity element is $e=1$. $e^{i\theta_1} \times e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ demonstrates closure, and the inverse of $e^{i\theta}$ is $e^{-i\theta}$ because $e^{i\theta}e^{-i\theta}=1$.

ii) This is not a group. 2 has no inverse: $2\times 0=0\neq 1$, $2\times 1=2\neq 1$, $2\times 2=0\neq 1$, $2\times 3=2\neq 1$.

iii) This is a group. The identity element is $M(0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

$$M(\theta_1)M(\theta_2)=\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}\begin{pmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{pmatrix}=\begin{pmatrix}\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2 & -(\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)\\ \sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\end{pmatrix}$$
$$=\begin{pmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{pmatrix}$$
$$=M(\theta_1+\theta_2),$$ so we have closure.

From the above we can also see that the inverse of $M(\theta_1)$ is $M(2\pi-\theta_1)$, as $M(\theta_1)M(2\pi-\theta_1)=M(2\pi)=M(0)$.

iv) This is a group. Clearly the identity element is $e=1$. The following Cayley table demonstrates closure and inverses.

| | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 7 |
| 3 | 3 | 1 | 7 | 5 |
| 5 | 5 | 7 | 1 | 3 |
| 7 | 7 | 5 | 3 | 1 |

so $1^{-1}=1$, $3^{-1}=3$, $5^{-1}=5$, $7^{-1}=7$.

v) This is not a group. The element $\begin{pmatrix}0&0\\0&0\end{pmatrix}$ has no inverse (as its determinant is zero).

vi) This is a group. Clearly the identity element is $e=1$. The following Cayley table demostrates closure and inverses.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 4 | 1 | 3 |
| 3 | 3 | 1 | 4 | 2 |
| 4 | 4 | 3 | 2 | 1 |

So $1^{-1}=1$, $2^{-1}=3$, $3^{-1}=2$, $4^{-1}=4$.

Clearly a finite and infinite group cannot be isomorphic.

(i) and (iii) are isomorphic. Consider the map $f$ defined by $e^{i\theta} \longrightarrow M(\theta)$.
Then $f(e^{i\theta_1} e^{i\theta_2}) = f(e^{i(\theta_1+\theta_2)}) = M(\theta_1+\theta_2) = M(\theta_1)M(\theta_2) = f(e^{i\theta_1})f(e^{i\theta_2})$, so $f$ is a homomorphism. $f$ is also clearly bijective, so $f$ is an isomorphism, so the two groups are isomorphic.

(iv) and (vi) are not isomorphic. In (iv), all elements are self-inverse but this is not the case in (vi).

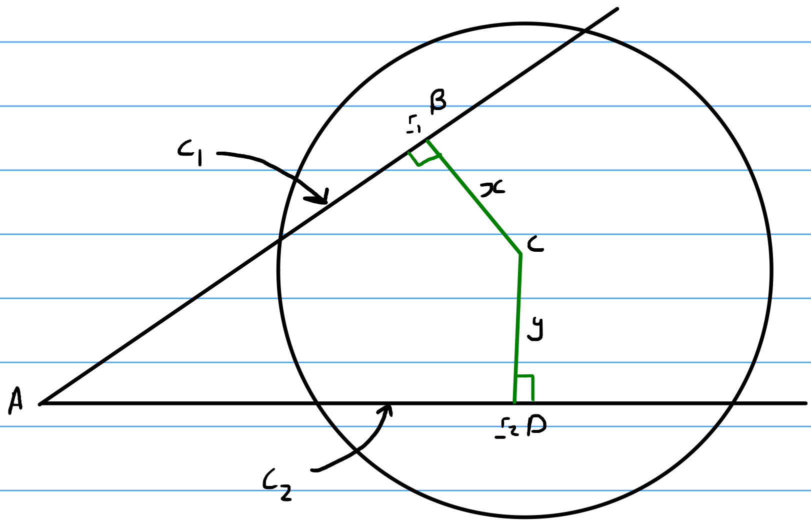
STEP III 1995 Q8

The equation of the normal to $\Pi$ through $X$ is $\underline{r} = \underline{x} + \lambda \underline{n}$.

We have $(\underline{x} + \lambda\underline{n}) \cdot \underline{n} = p$ when the line intersects the plane

$\Rightarrow \underline{x}\cdot\underline{n} + \lambda = p$

$\Rightarrow \lambda = p - \underline{x}\cdot\underline{n}$ is the perpendicular distance from $X$ to $\Pi$ (as $|\underline{n}| = 1$).




Both circles have the same (unit) radius, and so ABCD is a kite $\Rightarrow x = y$. Choose $\lambda$ such that the position vector of C is $\underline{r}_1 + \lambda \underline{n}_1$. Then the position vector of C is also $\underline{r}_2 \pm \lambda \underline{n}_2$, where the $\pm$ is because we don't know which direction $\underline{n}_2$ is pointing.

So we have $\underline{r}_1 + \lambda\underline{n}_1 = \underline{r}_2 \pm \lambda \underline{n}_2 \quad (*)$

$\Rightarrow \underline{r}_1 - \underline{r}_2 = \lambda(-\underline{n}_1 \pm \underline{n}_2)$

$$\Rightarrow (\underline{r}_1 - \underline{r}_2)\cdot(\underline{n}_1 \times \underline{n}_2) = \lambda(-\underline{n}_1 \pm \underline{n}_2)\cdot(\underline{n}_1\times\underline{n}_2)$$
$$= \lambda(-\underline{n}_1)\cdot(\underline{n}_1\times\underline{n}_2) \pm \lambda\underline{n}_2\cdot(\underline{n}_1\times\underline{n}_2)$$
$$= 0$$
, as required.

Geometrically, this says that the line of intersection of both planes (which has direction $\underline{n}_1 \times \underline{n}_2$) is perpendicular to the line joining $\underline{r}_1$ and $\underline{r}_2$.

Now, we dot $(*)$ with $\underline{n}_1$, so

$$\underline{r}_1 \cdot \underline{n}_1 + \lambda \underline{n}_1 \cdot \underline{n}_1 = \underline{r}_2 \cdot \underline{n}_1 \pm \lambda \underline{n}_2 \cdot \underline{n}_1$$

$$\Rightarrow p_1 + \lambda = \underline{r}_2 \cdot \underline{n}_1 \pm \lambda \underline{n}_1 \cdot \underline{n}_2$$

$$\Rightarrow p_1 - \underline{n}_1 \cdot \underline{r}_2 = \lambda(-1 \pm \underline{n}_1 \cdot \underline{n}_2)$$

Similarly dotting with $\underline{n}_2$,

$$\underline{r}_1 \cdot \underline{n}_2 + \lambda \underline{n}_1 \cdot \underline{n}_2 = \underline{r}_2 \cdot \underline{n}_2 \pm \lambda \underline{n}_2 \cdot \underline{n}_2$$

$$\Rightarrow \underline{r}_1 \cdot \underline{n}_2 + \lambda \underline{n}_1 \cdot \underline{n}_2 = p_2 \pm \lambda$$

$$\Rightarrow \lambda(\underline{n}_1 \cdot \underline{n}_2 \mp 1) = p_2 - \underline{r}_1 \cdot \underline{n}_2$$

Now, if these signs are $+$ then $-$, $p_1 - \underline{n}_1 \cdot \underline{r}_2 = \lambda(-1 + \underline{n}_1 \cdot \underline{n}_2)$,

$p_2 - \underline{r}_1 \cdot \underline{n}_2 = \lambda(\underline{n}_1 \cdot \underline{n}_2 - 1)$, so

$$(p_1 - \underline{n}_1 \cdot \underline{r}_2)^2 = (p_2 - \underline{r}_1 \cdot \underline{n}_2)^2,$$ as required.

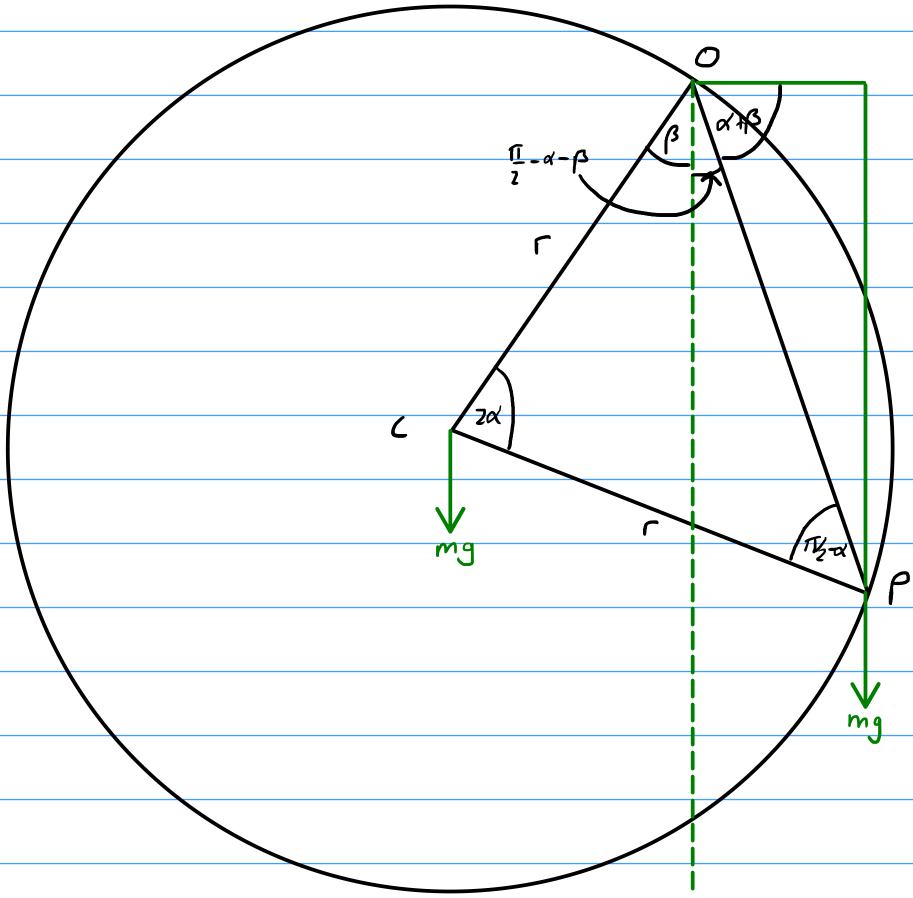
Similarly, if the signs are $-$ then $+$, $p_1 - \underline{n}_1 \cdot \underline{r}_2 = \lambda(-1 - \underline{n}_1 \cdot \underline{n}_2)$,

$p_2 - \underline{r}_1 \cdot \underline{n}_2 = \lambda(\underline{n}_1 \cdot \underline{n}_2 + 1)$, so

$$(p_1 - \underline{n}_1 \cdot \underline{r}_2)^2 = (p_2 - \underline{r}_1 \cdot \underline{n}_2)^2,$$ as required.

Geometrically, this says that the distance of $\underline{r}_1$ from $\Pi_2$ is the same as the distance of $\underline{r}_2$ from $\Pi_1$.

STEP III 1995 Q9




Note that as OCQ is isosceles, $OP = 2r\sin\alpha$.

i) Taking moments about O,

$$\cancel{m}\cancel{g}\cancel{r}\sin\beta = \cancel{m}\cancel{g}\cdot 2\cancel{r}\sin\alpha\cdot\cos(\alpha+\beta)$$

$$\Rightarrow \sin\beta = 2\sin\alpha(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

$$\Rightarrow \sin\beta(1+2\sin^2\alpha) = 2\sin\alpha\cos\alpha\cos\beta$$

$$\Rightarrow \tan\beta = \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}$$

$$= \frac{\sin 2\alpha}{1+(1-\cos 2\alpha)}$$

$$= \frac{\sin 2\alpha}{2-\cos 2\alpha},$$ as required.

ii) First consider the moment of inertia about C.

The total mass of the disc is $\int_0^r 2\pi x \cdot \frac{\rho x}{r}\, dx$

$$= \left[\frac{2}{3}\frac{\pi \rho x^3}{r}\right]_0^r$$

$$= \frac{2\pi}{3}\rho r^2 = m$$

$$\Rightarrow \rho = \frac{3m}{2\pi r^2}$$

so $I_c = \int_0^r \frac{\rho x}{r} \cdot 2\pi x \cdot x^2\, dx$

$$= \int_0^r \frac{3m}{2\pi r^2} \cdot \frac{2\pi}{r} \cdot x^4\, dx$$

$$= \frac{3m}{r^3}\left[\frac{1}{5}x^5\right]_0^r$$

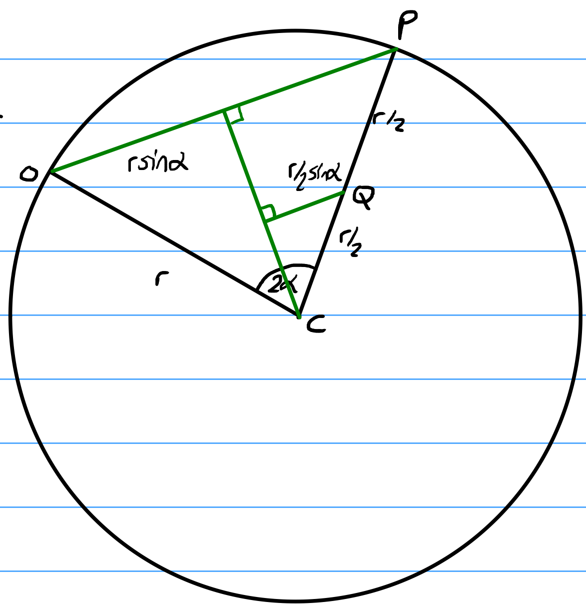
$$= \frac{3m}{5}r^2$$

By the parallel axis theorem,

$$I_0 = I_c + mr^2$$

$= \frac{8}{5}mr^2$, as required.

iii) When P is directly below O, point Q has fallen a total distance of $r\sin\alpha + \frac{r}{2}\sin\alpha = \frac{3r}{2}\sin\alpha$. Further, r is the centre of mass of the system, so $GPE_{loss} = 2mg \cdot \frac{3r}{2}\sin\alpha = 3mgr\sin\alpha$.




So the KE of the system is $3mgr\sin\alpha$.

Now, the total moment of inertia about O is

$\frac{8}{5}mr^2 + m(2r\sin\alpha)^2$ (as $OP = 2r\sin\alpha$)

$= \frac{8}{5}mr^2 + 4mr^2\sin^2\alpha$

$= \frac{4}{5}mr^2(2 + 5\sin^2\alpha)$.

Now $KE = \frac{1}{2}I\omega^2$

$$\Rightarrow 3\cancel{m}g\cancel{r}\sin\alpha = \frac{1}{2}\cdot\frac{4}{5}\cancel{m}r^{\cancel{2}}(2+5\sin^2\alpha)\omega^2$$

$$\Rightarrow \omega^2 = \frac{15g\sin\alpha}{2r(2+5\sin^2\alpha)}$$

C is a distance r from O, so

$$v^2 = r^2\omega^2 = \frac{15gr\sin\alpha}{2(2+5\sin^2\alpha)}$$ , as required.

Now $\frac{dv^2}{d\alpha} = \frac{30gr\cos\alpha(2+5\sin^2\alpha) - 15gr\sin\alpha \times 20\sin\alpha\cos\alpha}{4(2+5\sin^2\alpha)} = 0$

$$\Rightarrow 60gr\cos\alpha + 150gr\sin^2\alpha\cos\alpha - 300gr\sin^2\alpha\cos\alpha = 0$$

$$\Rightarrow \cos\alpha(60-150\sin^2\alpha) = 0$$

$$\Rightarrow \cos\alpha = 0 \quad \text{or} \quad \sin\alpha = \sqrt{\frac{2}{5}}$$

when $\cos\alpha = 0$, $\sin\alpha = 1$

$$\Rightarrow v^2 = \frac{15gr}{14} \approx 1.07gr$$

when $\sin\alpha = \sqrt{\frac{2}{5}}$,

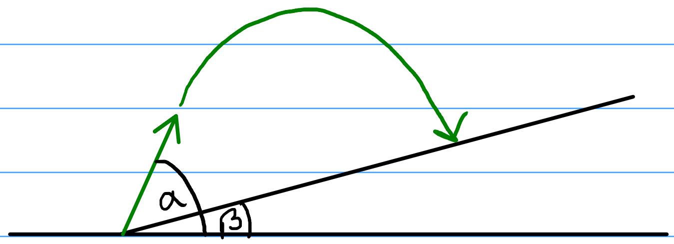
$$v^2 = \frac{15gr}{2}\cdot\sqrt{\frac{2}{5}}\cdot\frac{1}{4}$$

$$= \frac{15gr\sqrt{10}}{40}$$

$$= \frac{3gr\sqrt{10}}{8} \approx 1.19gr$$

So $\sin\alpha = \sqrt{\frac{2}{5}}$ gives the maximum of $v^2 = \frac{3gr\sqrt{10}}{8}$.

# STEP III 1995 Q10




Horizontally $x = ut\cos\alpha$ Vertically, $y = ut\sin\alpha - \frac{1}{2}gt^2$

The particle hits the plane when $\frac{y}{x} = \tan\beta \Rightarrow y = x\tan\beta$

$$\text{so } ut\sin\alpha - \tfrac{1}{2}gt^2 = ut\cos\alpha\tan\beta$$

$$\Rightarrow t\left(\tfrac{1}{2}gt + u(\cos\alpha\tan\beta - \sin\alpha)\right) = 0$$

$t \neq 0$, so
$$t = \frac{2u}{g}(\sin\alpha - \cos\alpha\tan\beta)$$
$$= \frac{2u}{g\cos\beta}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \frac{2u}{g\cos\beta}\sin(\alpha - \beta).$$

Maximising the distance up the plane is equivalent to maximising the $x$ coordinate at the time of collision with the plane.

So we want to maximise $x = ut\cos\alpha$
$$= \frac{2u^2}{g\cos\beta}\sin(\alpha-\beta)\cos\alpha$$
$$= \frac{u^2}{g\cos\beta}\left(\sin(2\alpha - \beta) - \sin\beta\right)$$

So we must have $2\alpha - \beta = \pi/2 \Rightarrow \alpha = \pi/4 + \beta/2$.

Now
$$d = \frac{x}{\cos\beta} = \frac{u^2}{g\cos^2\beta}(1 - \sin\beta)$$
$$= \frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)}$$
$$= \frac{u^2(1-\sin\beta)}{g(1+\sin\beta)(1-\sin\beta)}$$
$$= \frac{u^2}{g(1+\sin\beta)}, \text{ as required.}$$

STEP III 1995 Q11

On the outword journey, we must have the velocity of the plane relative to the ship as $\binom{-u}{u}$. Further, the velocity of the ship relative to the air is $\binom{-v}{v}$. So, the velocity of the plane relative to the air is $\binom{-u-v}{u+v}$. As the plane has airspeed $kV$, we must have $2(u+v)^2 = k^2v^2$ $\Rightarrow \sqrt{2}(u+v) = kV \Rightarrow u_{out} = \frac{(k-\sqrt{2})}{\sqrt{2}}V$. Similarly, on the return journey the velocity of the plane relative to the ship is $\binom{u}{-u}$, and the velocity of the ship relative to the air is still $\binom{-v}{v}$. So, the velocity of the plane relative to the air is $\binom{u-v}{-u+v}$, so $2(u-v)^2 = k^2v^2 \Rightarrow \sqrt{2}(u-v) = kV$ $\Rightarrow u_{back} = \frac{k+\sqrt{2}}{\sqrt{2}}V$.

As time is inversely proportional to speed, the fraction of time spent on the outward journey is

$$\frac{u_{back}}{u_{out}+u_{back}} = \frac{\frac{k+\sqrt{2}}{\sqrt{2}}V}{\frac{k-\sqrt{2}}{\sqrt{2}}V + \frac{k+\sqrt{2}}{\sqrt{2}}V} = \frac{k+\sqrt{2}}{2k}.$$

As the whole journey takes 0.5 hours, the outward flight lasts $\frac{k+\sqrt{2}}{4k}$ hours.

Note that the furthest displacement of the plane from the ship is

$$\frac{k-\sqrt{2}}{\sqrt{2}}V \cdot \frac{k+\sqrt{2}}{4k}\binom{1}{-1}$$

$$= \frac{k^2-2}{4\sqrt{2}k}V\binom{1}{-1}$$

so $d_w = \sqrt{2}\,\frac{k^2-2}{4\sqrt{2}k}V = \frac{k^2-2}{4k}V$ (as $\left|\binom{-1}{1}\right| = \sqrt{2}$)

Now, with the wind calm, on the outward journey the velocity of the plane relative to the ship is $\binom{-u}{u}$ and of the ship relative to the air is $\binom{-v}{0}$, so the velocity of the plane relative to the air is $\binom{-u-v}{u}$.

So, $(u+v)^2 + u^2 = k^2 v^2$

$\Rightarrow 2u^2 + 2uv + (1-k^2)v^2$

$$\Rightarrow u_{out} = \frac{-2v + \sqrt{4v^2 - 8(1-k^2)v^2}}{4}$$

$$= \frac{-v + v\sqrt{2k^2-1}}{2}$$

$$= \frac{v}{2}\left(\sqrt{2k^2-1} - 1\right).$$

On the return journey, the same argument follows, but the first line is $(u-v)^2 + u^2 = k^2v^2$

$$\Rightarrow u_{back} = \frac{v}{2}\left(\sqrt{2k^2-1} + 1\right)$$

As before, we have

$$t_c = \frac{1}{2} \cdot \frac{u_{back}}{u_{out} + u_{back}} = \frac{1}{2} \cdot \frac{\frac{v}{2}(\sqrt{2k^2-1}+1)}{\frac{v}{2}(\sqrt{2k^2-1}-1) + \frac{v}{2}(\sqrt{2k^2-1}+1)}$$

$$= \frac{\sqrt{2k^2-1} + 1}{4\sqrt{2k^2-1}}$$

$$\text{so } d_c = \sqrt{2} \cdot \frac{\sqrt{2k^2-1}+1}{4\sqrt{2k^2-1}} \cdot \frac{v}{2} \cdot \left(\sqrt{2k^2-1} - 1\right)$$

$$= \frac{\sqrt{2}v(2k^2-2)}{8\sqrt{2k^2-1}}$$

$$= \frac{\sqrt{2}v(k^2-1)}{4\sqrt{2k^2-1}}$$

$$\text{So } \frac{d_w}{d_c} = \frac{v(k^2-2)}{4k} \cdot \frac{4\sqrt{2k^2-1}}{\sqrt{2}v(k^2-1)}$$

$$= \frac{k^2-2}{2k(k^2-1)} \cdot \sqrt{2}\sqrt{2k^2-1}$$

$$= \frac{k^2-2}{2k(k^2-1)} \cdot \sqrt{4k^2-2},$$ as required.

STEP III 1995 Q12

$f(x,y) = f(x)f(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}$

$= \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}$

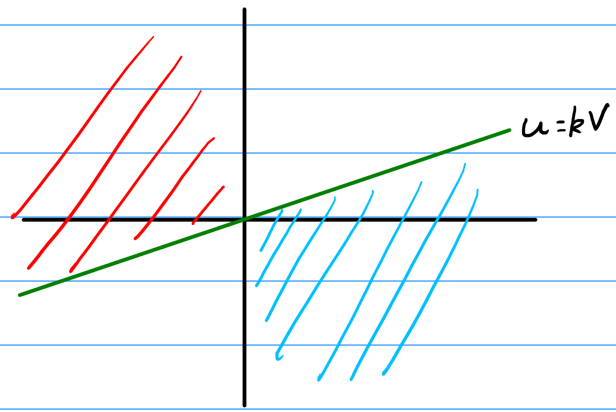
This density is radially symmetrical because it depends on $x$ and $y$ only through $x^2+y^2$, or in other words it depends only on the distance from the origin.

$P(\frac{u}{v} < k) = P(\frac{u}{v} < k, v<0) + P(\frac{u}{v} < k, v>0)$

$= P(u > kv, v<0) + P(u<kv, v>0)$

$=$ red area + blue area

Note the red area is just a rotation of the blue area by $\pi$. Because the density is radially symmetric, the probabilities of both regions is equal.



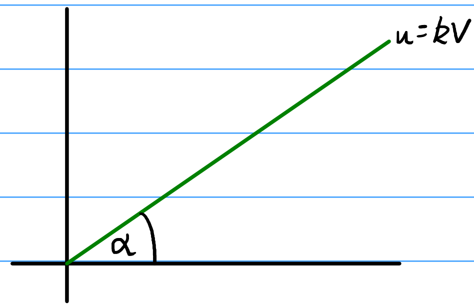

So $P(\frac{u}{v} < k) = 2P(u<kv, v>0)$.

Now $P(\frac{u}{v} < k) = 2P(u<kv, v>0)$

$= 2\left(\frac{1}{4} + \frac{\alpha}{2\pi}\right)$ (because $g$ is radially symmetrical)

$= \frac{1}{2} + \frac{\alpha}{\pi}$ (this still works for $k<0$, as then $\alpha < 0$)

$= \frac{1}{2} + \frac{\arctan k}{\pi}$

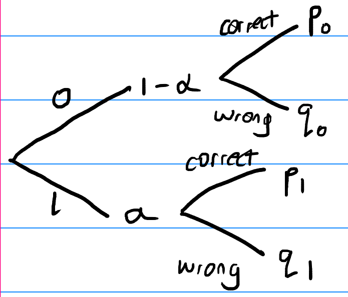



So $f(k) = \frac{d}{dk}\left(\frac{1}{2} + \frac{\arctan k}{\pi}\right)$

$= \frac{1}{\pi(1+x^2)}$, as required.

STEP III 1995 Q1

$P(\text{all digits correct}) = p_1^{\alpha 10^k} \, p_2^{(1-\alpha)10^k}$




$$\beta = P(1) = (1-\alpha)q_0 + p_1\alpha$$
$$\beta = \alpha \iff \alpha = (1-\alpha)q_0 + (1-q_1)\alpha$$
$$\iff \alpha(1+q_0-1+q_1) = q_0$$
$$\iff \alpha = \frac{q_0}{q_0+q_1}.$$

Now $P(\text{sent } 0 \mid \text{recieved } 0) = \dfrac{P(\text{recieved } 0 \mid \text{sent } 0)P(\text{sent } 0)}{P(\text{recieved } 0)}$

$$= \frac{p_0(1-\alpha)}{p_0(1-\alpha)+(1-p_1)\alpha}$$

$$= \frac{p_0 \cdot \frac{q_1}{q_0+q_1}}{p_0\frac{q_1}{q_0+q_1} + q_1\frac{q_0}{q_0+q_1}}$$

$$= \frac{p_0}{p_0+q_0}$$

$$= p_0.$$

So $P(\text{entire message correct}) = p_0^{10^k}$.

Now $q_0 = q_1 = q$ and $\alpha = 1/2$.

$P(\text{all recieved correctly}) = p^{10^6} = 1/2$
$$\Rightarrow 10^6 \ln(p) = \ln(\tfrac{1}{2})$$
$$\Rightarrow \ln(p) = \frac{\ln(1/2)}{10^6}$$
$$\Rightarrow p = \exp\left(\frac{\ln(1/2)}{10^6}\right)$$

$$\Rightarrow q = 1 - p$$
$$= 1 - \exp\left(\frac{1}{10^6} \times \ln\left(\frac{1}{2}\right)\right)$$
$$\approx 1 - \left(1 + \frac{1}{10^6} \times \ln\left(\frac{1}{2}\right)\right)$$
$$= \frac{-1}{10^6} \times \ln\left(\frac{1}{2}\right) \approx 6.93 \times 10^{-7}$$

Now, we have a booster. Note that the probability of a digit being recieved correctly after two errors (which cancel each other out) is of order $q^2$, which we can ignore. Further, as the cable is half as long, the probability of each error is reduced by a factor of 4.

So, $P(\text{all correctly received}) = \left(1 - q/4\right)^{10^6} \left(1 - q/4\right)^{10^6}$
$$= \left[\left(1 + \ln\left(\frac{1}{2}\right) \times \frac{1}{4 \times 10^6}\right)^{10^6}\right]^2$$
$$\approx \left(1 + \frac{1}{4} \ln \frac{1}{2}\right)^2$$
$$\approx 0.683$$

STEP III 1995 Q14

$$M_T(\theta) = Ee^{\theta T} = \int_0^\infty e^{\theta t} te^{-t} \, dt$$ this only exists for $\theta < 1$

$$= \int_0^\infty te^{(\theta-1)} dt$$

$u \quad t \qquad v' \quad e^{(\theta-1)t}$

$u' \quad 1 \qquad v \quad \frac{1}{\theta-1}e^{(\theta-1)t}$

$$= \left[\frac{t}{\theta-1} e^{(\theta-1)t}\right]_0^\infty - \frac{1}{\theta-1}\int_0^\infty e^{(\theta-1)t} dt$$

$$= \quad 0 \quad - \frac{1}{(\theta-1)^2}\left[e^{(\theta-1)t}\right]_0^\infty$$

$$= \frac{1}{(\theta-1)^2}$$

$$M_{2T}(\theta) = M_T(\theta)^2 = \frac{1}{(\theta-1)^4}$$

Now if $U$ has density $\frac{1}{6}u^3e^{-u}$, then

$$M_u(\theta) = \int_0^\infty \frac{1}{6}e^{\theta u} u^3 e^{-u} du$$

$$= \int_0^\infty \frac{1}{6}u^3 e^{(\theta-1)u} du$$

$u \quad \frac{1}{6}u^3 \qquad v' \quad e^{(\theta-1)u}$

$u' \quad \frac{1}{2}u^2 \qquad v \quad \frac{1}{\theta-1}e^{(\theta-1)u}$

$$= \left[\frac{1}{6(\theta-1)}u^3e^{(\theta-1)u}\right]_0^\infty - \frac{1}{\theta-1}\int_0 \frac{1}{2}u^2e^{(\theta-1)u} du$$

$$= \quad 0 \quad - \frac{1}{\theta-1}\int_0^\infty \frac{1}{2}u^2e^{(\theta-1)u} du$$

$u \quad \frac{1}{2}u^2 \qquad v' \quad e^{(\theta-1)u}$

$u' \quad u \qquad v \quad \frac{1}{\theta-1}e^{(\theta-1)u}$

$$= \frac{-1}{\theta-1}\left[\frac{1}{2(\theta-1)}u^2e^{(\theta-1)u}\right]_0^\infty + \frac{1}{(\theta-1)^2}\int_0^\infty ue^{(\theta-1)u} du$$

$$= 0 + \frac{1}{(\theta-1)^2}\cdot\frac{1}{(\theta-1)^2}$$

$$= \frac{1}{(\theta-1)^4}$$. So $M_u(\theta) = M_{2T}(\theta) \Rightarrow U \stackrel{d}{=} 2T$, as required.

Now, we claim that $nY$ has density $\frac{1}{(2n-1)!} \cdot t^{2n-1} e^{-t}$ and $M_{nY}(\theta) = \frac{1}{(\theta-1)^{2n}}$. We prove this via induction. The base case is clear. Assume it is true for $n=k+1$. Then, for $n=k+1$,

$$M_{(k+1)Y}(\theta) = M_{kY}(\theta)M_Y(\theta)$$
$$= \frac{1}{(\theta-1)^{2k}} \cdot \frac{1}{(\theta-1)^2}$$
$$= \frac{1}{(\theta-1)^{2k+2}}$$

Further, consider $U$ with density $\frac{1}{(2k+1)!} u^{2k+1} e^{-t}$.

Then $M_U(\theta) = \frac{1}{(2k+1)!} \int_0^\infty u^{2k+1} e^{(\theta-1)u} \, du$

$u \quad u^{2k+1} \qquad v' \quad e^{(\theta-1)u}$

$u' \quad (2k+1)u^{2k} \qquad v \quad \frac{1}{\theta-1} e^{(\theta-1)u}$

$$= \frac{1}{(2k+1)!}\left[\frac{1}{\theta-1} u^{2k+1} e^{(\theta-1)u}\right]_0^\infty - \frac{1}{(2k)!} \cdot \frac{1}{\theta-1} \int_0^\infty u^{2k} e^{(\theta-1)u} \, du$$

$$= \frac{-1}{(2k)!} \cdot \frac{1}{\theta-1} \int_0^\infty u^{2k} e^{(\theta-1)u} \, du$$

$u \quad u^{2k} \qquad v' \quad e^{(\theta-1)u}$

$u' \quad 2ku^{2k-1} \qquad v \quad \frac{1}{\theta-1} e^{(\theta-1)u}$

$$= \frac{-1}{(2k)!(\theta-1)}\left[\frac{1}{\theta-1} u^{2k} e^{(\theta-1)u}\right]_0^\infty + \frac{1}{(\theta-1)^2} \cdot \int_0^\infty \frac{1}{(2k-1)!} \cdot u^{2k-1} e^{(\theta-1)u} \, du$$

$$= 0 + \frac{1}{(\theta-1)^2} \cdot \frac{1}{(\theta-1)^{2k}} \quad \text{by assumption}$$
$$= \frac{1}{(\theta-1)^{2k+2}}$$
$$= M_{(k+1)Y}(\theta)$$

so $U \overset{d}{=} (k+1)Y$.

So, by induction, $nY$ has density $\frac{1}{(2n-1)!} \cdot t^{2n-1} e^{-t}$.