STEP II 1995 Q1

i) $(1+x+x^2+ \text{——} +x^n)(1-x) = 1+x+x^2+ \text{——} +x^n - x - x^2 - \text{——} - x^{n+1}$

$= 1-x^{n+1}$

So $(1+x+x^2+ \text{——} +x^n) = \dfrac{1-x^{n+1}}{1-x}$

ii) Differentiating both sides w.r.t x,

$$1+2x+3x^2+ \text{——} + nx^{n-1} = \frac{(1-x)(-(n+1)x^n)+(1-x^{n+1})}{(1-x)^2}$$

Setting $x=-1$,

$$1-2+3-4+ \text{——} +(-1)^{n-1}n = \frac{-2(n+1)(-1)^{n+1}+1-(-1)^{n+1}}{4}$$

If n is even, $= \dfrac{-2(n+1)+1+1}{4} = \dfrac{-n}{2}$

If n is odd, $= \dfrac{2(n+1)+1-1}{4} = \dfrac{n+1}{2}$

iii) $1^2-2^2+3^2-4^2+ \text{——} +(-1)^{n-1}n^2$

If n is even,

$= (1^2-2^2)+(3^2-4^2)+ \text{——} +((n-1)^2-n^2)$

$= (1+2)(1-2)+(3+4)(3-4)+ \text{——} +(n-1+n)(n-1-n)$

$= -(3+7+ \text{——} + 2n-1)$

$= -\sum_{r=1}^{n/2} 4r-1$

$= -\left(4\frac{n}{4}\left(\frac{n}{2}+1\right) - n/2\right)$

$= -\left(\frac{n^2}{2}+n-\frac{n}{2}\right)$

$= -\left(\frac{n^2}{2}+\frac{n}{2}\right)$

$= (-1)^{n-1}\left(\frac{1}{2}n^2+\frac{1}{2}n\right)$

If n is odd,

$= 1^2 - 2^2 + \cdots + (n-2)^2 - (n-1)^2 + n^2$

$= -\frac{(n-1)}{2} - \frac{(n-1)^2}{2} + n^2$ (by the previous result)

$= \frac{1}{2}(-n+1-n^2+2n-1+2n^2)$

$= \frac{1}{2}(n^2+n)$

$= (-1)^n\left(\frac{1}{2}n^2+\frac{1}{2}n\right)$

So $A = B = \frac{1}{2}$ in both cases.

STEP II 1995 Q2

If I have $n+1$ posts and want to paint the first and $(n+1)^{th}$ red, this is equivalent to having n posts, painting the first red and the n^{th} not red, then the $(n+1)^{th}$ post red.

So, $r_{n+1} = s_n$.

Now, $r_n + s_n$ is the number of ways of painting all fenceposts differently from their neighbours, with the first post red. Thus after the first post we have exactly 2 choices for each further post depending on the previous post.

So $r_n + s_n = 2^{n-1}$

Claim: $r_n = \dfrac{2^{n-1} + 2(-1)^{n-1}}{3}$

Proof:
For $n=1$, clearly $r_n = 1$, $\dfrac{2^0 + 2(-1)^0}{3} = \dfrac{1+2}{3} = 1$, so true for $n=1$.

Assume true for $n=k$.

Then for $n=k+1$, $r_{k+1} = s_k = 2^{k-1} - r_k$

$$= 2^{k-1} - \frac{2^{k-1} + 2(-1)^{k-1}}{3}$$

$$= \frac{3(2^{k-1}) - 2^{k-1} - 2(-1)^{k-1}}{3}$$

$$= \frac{2(2^{k-1}) + 2(-1)^k}{3}$$

$$= \frac{2^k + 2(-1)^k}{3}$$

True for $n=1$, and if true for $n=k$ then true for $n=k+1$, so true for all $n \in \mathbb{N}$ by induction.

If we have $(n+1)$ fenceposts, with the first and $(n+1)^{th}$ red, we can join them in a circle, and delete the final red post to get a circle of n posts that work. There are r_{n+1} ways of doing this. But we could have also started with white or blue.

So the number of ways is $3r_{n+1} = 2^n + 2(-1)^n$

STEP II 1995 Q3

Stage	A	B	C	D	E
1	5				
2	5				
3	5				
4	5				3
5	4				5

A scored 24 points overall, so got 5 points 4 times, and 4 points once. E scored 5 points in the final race, so A must have scored 5, 5, 5, 5, 4.

In total, there are 75 points available. A scored 24, so $75 - 24 = 51$, so B, C, D, E got 51 points between them. $\frac{51}{4} = 12.75$. E must have got less than average, so $E \leq 12$. But if $E = 12$, the minimum possible scores of D, C, and B are 13, 14, and 15. But $12+13+14+15 = 54 > 51$. So must have $E = 11$.

So between them, B, C, and D scored $51 - 11 = 40$, with each score ≥ 12. Note $12+13+15 = 40$, and no other combinations work (can't make C or D lower, as all distinct, and if B becomes 14, then either $D = 13$ or $C = 14$, a contradiction).

So $B = 15$, $C = 13$, $D = 12$.

C got 4 of the same score. Can't get 4 lots of 5 or 4 (as total too high), and can't get 4 lots of 1 (as E already has 3 of them). If C got 4 lots of 2, $4 \times 2 = 8$, so C must have got a 5. But all the 5s are already accounted for. So C got 4 lots of 3 and a 1. The 3 in the fourth race is already accounted for, so C got 3, 3, 3, 1, 3.

Now, left to put in the grid are 1, 2222, 4444. B got 15 points, an odd number, so must have got the 1. The only 1 not accounted for is in the final stage. So Borthes came 5^{th} in the final stage, scoring one point.

STEP II 1995 Q4

$u_n = \int_0^{\pi/2} \sin^n t \, dt$

$u \quad \sin^{n-1} t \qquad v' \quad \sin t$

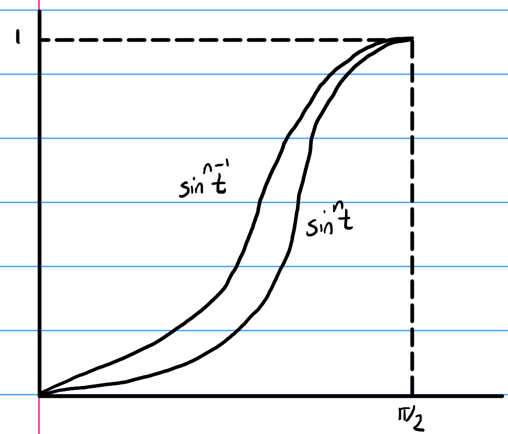
$u' \quad (n-1)\sin^{n-2} t \cos t \qquad v \quad -\cos t$

$$u_n = \left[-\cos t \sin^{n-1} t\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 t \sin^{n-2} t$$
$$= 0 + (n-1)\int_0^{\pi/2} (1-\sin^2 t)\sin^{n-2} t$$
$$= (n-1)(u_{n-2} - u_n)$$

$\Rightarrow nu_n = (n-1)u_{n-2}$

So $nu_n u_{n-1} = (n-1)u_{n-1}u_{n-2}$

So $nu_n u_{n-1} = (n-1)u_{n-1}u_{n-2}$
$$= (n-2)u_{n-2}u_{n-3}$$
$$= \dots$$
$$= u_1 u_0$$
$$= \int_0^{\pi/2} \sin t \, dt \cdot \int_0^{\pi/2} dt$$
$$= \left[-\cos t\right]_0^{\pi/2} \cdot \pi/2$$
$$= (0+1)\frac{\pi}{2}$$
$$= \pi/2, \text{ as required.}$$

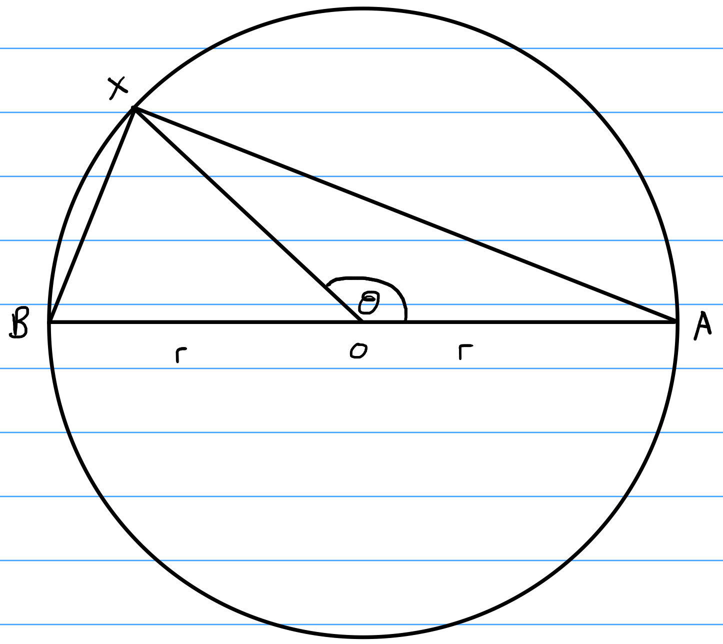


$\sin^{n-1} t > \sin^n t$ for $0 < t < \pi/2$, so $u_{n-1} > u_n$, and $\sin^n t > 0$ for $0 < t < \pi/2$, so $u_n > 0$. So $0 < u_n < u_{n-1}$.

$nu_n^2 < nu_n u_{n-1} = \pi/2$, and $nu_{n-1}^2 > nu_n u_{n-1} = \pi/2$.
So $nu_n^2 < \pi/2 < nu_{n-1}^2$.

We also have $(n+1)u_{n+1}^2 < \frac{\pi}{2} < (n+1)u_n^2$, so $(n+1)u_n^2 > \frac{\pi}{2} \Rightarrow nu_n^2 > \left(\frac{n}{n+1}\right)\frac{\pi}{2}$. So $\left(\frac{n}{n+1}\right)\frac{\pi}{2} < nu_n^2 < \pi/2$, as required. As $n \longrightarrow \infty$, $\frac{n}{n+1} \longrightarrow \pi/2$, so we must have $nu_n^2 \longrightarrow \pi/2$.

STEP II 1995 Q5



Assume swim speed is v, and run speed is kv.

Swimming takes $\frac{2r}{v}$, running takes $\frac{\pi r}{kv}$. So swimming is faster than running if

$$\frac{2r}{v} < \frac{\pi r}{kv} \iff k < \frac{\pi}{2}.$$

For option c, the run distance is $r\theta$, and the swim distance is $\sqrt{2r^2 - 2r^2\cos(\pi-\theta)}$ by the cosine rule. This is $r\sqrt{2+2\cos\theta}$

$$= r\sqrt{2 + 4\cos^2\frac{\theta}{2} - 2}$$

$$= 2r\cos\frac{\theta}{2}.$$

So time taken is $\frac{r}{v}\left(2\cos\frac{\theta}{2} - \frac{\theta}{k}\right)$

$\frac{dT}{d\theta} = \frac{r}{v}\left(-\sin\frac{\theta}{2} + k\right)$. Note $\sin\frac{\theta}{2}$ is increasing for $0<\theta<\pi$, so $\frac{dT}{d\theta}$ is decreasing. Thus any extrema of T are local maxima. So the minimum value for T occurs at $\theta = 0$ or $\theta = \pi$, which is just option option a or option b.

So, we swim if $k \leq \pi/2$, run if $k \geq \pi/2$, and never do both. Note equality holds when $k = \pi/2$.

STEP II 1995 Q6

$(z-u)(z-v) = z^2-(u+v)z+uv \Rightarrow a=-u-v$ and $b=uv$

$\alpha = \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$ so $\alpha^7 = \cos\frac{14\pi}{7}+i\sin\frac{14\pi}{7}$ (by de Moivre)

$= 1 + 0i$

$= 1$

So $\alpha^7-1 = 1-1=0$, so α is a root of z^7-1. The other roots are $\alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7(=1)$, as these are the 7^{th} roots of unity.

$\alpha+\alpha^2+\alpha^4$ is a root of z^2+Az+B

$(\alpha+\alpha^2+\alpha^4)^* = \alpha^*+\alpha^{2*}+\alpha^{4*}$

$= \alpha^{-1}+\alpha^{-2}+\alpha^{-4}$

$= \alpha^6+\alpha^5+\alpha^3$, so this is also a root as A and B are real.

$(\alpha+\alpha^2+\alpha^4)^2 + A(\alpha+\alpha^2+\alpha^4)+B=0$

$\Rightarrow \alpha^2+\alpha^4+\alpha^8+2\alpha^3+2\alpha^5+2\alpha^6+A\alpha+A\alpha^2+A\alpha^4+B=0$

$\Rightarrow \alpha^2+\alpha^4+\alpha+2\alpha^3+2\alpha^5+2\alpha^6+A\alpha+A\alpha^2+A\alpha^4+B=0$

$\Rightarrow 2\alpha^6+2\alpha^5+(A+1)\alpha^4+2\alpha^3+(A+1)\alpha^2+(A+1)\alpha+B=0$ (1)

$(\alpha^3+\alpha^5+\alpha^6)^2+A(\alpha^3+\alpha^5+\alpha^6)+B=0$

$\Rightarrow \alpha^6+\alpha^{10}+\alpha^{12}+2\alpha^8+2\alpha^9+2\alpha^{11}+A\alpha^3+A\alpha^5+A\alpha^6+B=0$

$\Rightarrow \alpha^6+\alpha^3+\alpha^5+2\alpha+2\alpha^2+2\alpha^4+A\alpha^3+A\alpha^5+A\alpha^6+B=0$

$\Rightarrow (A+1)\alpha^6+(A+1)\alpha^5+2\alpha^4+(A+1)\alpha^3+2\alpha^2+2\alpha+B=0$ (2)

Note that because $\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6=-1$, $A=1$ and $B=-2$ are solutions.

The roots are conjugates of each other, so $-(u+v)$

$= -2\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right) = 1$

$\Rightarrow \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7} = -\frac{1}{2}$.

Similarly, $uv = \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right)^2 + \left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)^2 = 2$

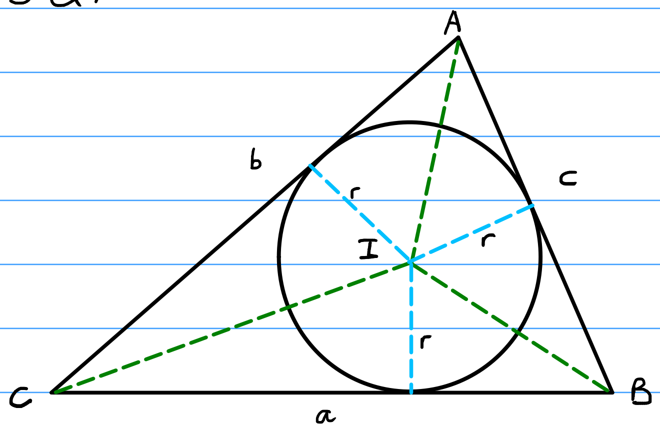
$$\Rightarrow \quad \frac{1}{4} + \left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)^2 = 2$$

$$\Rightarrow \left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)^2 = 7/4$$

$$\Rightarrow \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \sqrt{7}/2$$

We take the positive square root because $\left|\sin\frac{2\pi}{7}\right| > \left|\sin\frac{8\pi}{7}\right|$, and $\sin\frac{2\pi}{7}, \sin\frac{4\pi}{7} > 0$.

STEP II 1995 Q7



i) $\Delta = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc$

$= \frac{1}{2}r(a+b+c)$

$= rs$

ii) $\Delta = \frac{1}{2}bc\sin\alpha$

$\Delta^2 = \frac{1}{4}b^2c^2\sin^2\alpha$

$= \frac{1}{4}b^2c^2(1-\cos^2\alpha)$

$= \frac{1}{4}b^2c^2 - \frac{1}{4}b^2c^2\cos^2\alpha$

$= \frac{1}{16}\left(4b^2c^2 - (2bc\cos\alpha)^2\right)$

Now $2bc\cos\alpha = b^2+c^2-a^2$

So $\Delta^2 = \frac{1}{16}\left(4b^2c^2 - (b^2+c^2-a^2)^2\right)$

$= \frac{1}{16}\left(4b^2c^2 - a^4 - b^4 - c^4 + 2a^2c^2 + 2b^2c^2 - 2b^2c^2\right)$

$= \frac{1}{16}\left(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\right)$

and $(a^2-(b-c)^2)((b+c)^2-a^2)$

$= (a^2-b^2-c^2+2bc)(b^2+c^2+2bc-a^2)$

$= a^2b^2 + a^2c^2 + \cancel{2a^2bc} - a^4 - b^4 - b^2c^2 - \cancel{2b^3c} + a^2b^2 - c^2b^2 - c^4 - \cancel{2bc^3} + a^2c^2 + \cancel{2b^3c} + \cancel{2bc^3} + 4b^2c^2$
$\cancel{-2a^2bc}$

$= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$

So $\Delta^2 = \frac{1}{16}(a^2-(b-c)^2)((b+c)^2-a^2)$

Further, $s(s-a)(s-b)(s-c)$

$$= \frac{1}{16}(a+b+c)(-a+b+c)(-b+a+c)(-c+a+b)$$

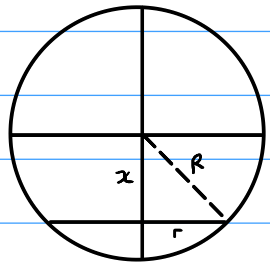
$$= \frac{1}{16}\big((b+c)+a\big)\big((b+c)-a\big)\big(a+(c-b)\big)\big(a-(c-b)\big)$$

$$= \frac{1}{16}\big((b+c)^2-a^2\big)\big(a^2-(b-c)^2\big)$$

$$= \Delta^2$$

So, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, as required.

iii)



We have $x = \sqrt{R^2 - r^2}$

but $\Delta = rs \Rightarrow r = \frac{\Delta}{s} \Rightarrow r^2 = \frac{\Delta^2}{s^2}$

so $x = \sqrt{R^2 - \frac{1}{s}(s-a)(s-b)(s-c)}$

STEP II 1995 Q8

$$\frac{dx}{dt} = (2k - x)x$$

$$\Rightarrow \frac{1}{x(2k-x)}dx = dt$$

Now $\frac{1}{x(2k-x)} \equiv \frac{A}{x} + \frac{B}{2k-x}$

$$\Rightarrow 1 \equiv A(2k-x) + Bx$$

$$x = 0 \Rightarrow A = \frac{1}{2k}$$

$$x = 2k \Rightarrow B = \frac{1}{2k}$$

So $\frac{1}{2k}\int \frac{1}{x} + \frac{1}{2k-x}\, dx = \int dt$

$$\Rightarrow \frac{1}{2k}\left(\ln|x| - \ln|2k-x|\right) = t + c_1$$

$$\Rightarrow \ln|x| - \ln|2k-x| = 2kt + c_2$$

$$\Rightarrow \ln\left|\frac{x}{2k-x}\right| = 2kt + c_2$$

$$\Rightarrow \frac{x}{2k-x} = Ae^{2kt}$$

$x(0) = k \Rightarrow \frac{k}{2k-k} = A \Rightarrow A = 1$

So $\frac{x}{2k-x} = e^{2kt}$

$$\Rightarrow x = 2ke^{2kt} - xe^{2kt}$$

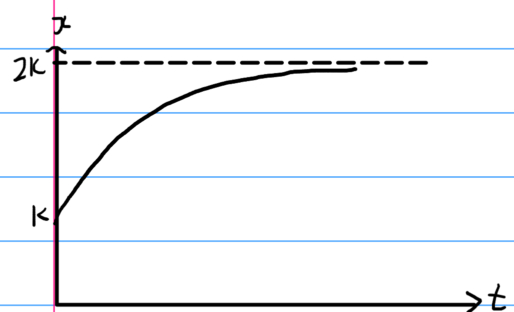
$$\Rightarrow xe^{-2kt} = 2k - x$$

$$\Rightarrow x(1 + e^{-2kt}) = 2k$$

$$\Rightarrow x = \frac{2k}{1+e^{-2kt}} = \frac{k + ke^{-2kt} - k - ke^{-2kt} + 2k}{1+e^{-2kt}}$$

$$= k + k\frac{1-e^{-2kt}}{1+e^{-2kt}}, \text{ as required.}$$

Note $x(0) = k$ and $\lim_{t\to\infty} x = 2k$, so



Now $\frac{dx}{dt} = (2k - x)x - L$

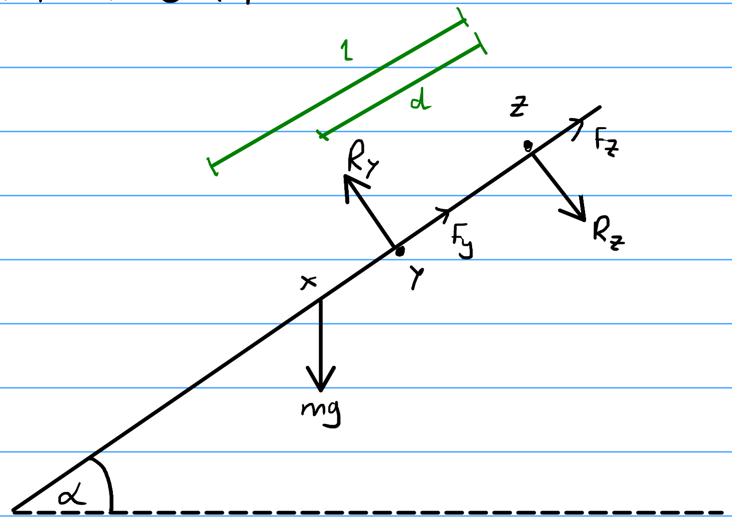
setting $y = x - k + \alpha$, where $\alpha = \sqrt{k^2 - L}$, we have $dy = dx$, so

$$\begin{aligned}
\frac{dy}{dt} &= (2k - (y + k - \alpha))(y + k - \alpha) - L \\
&= (k - y + \alpha)(k + y - \alpha) - L \\
&= (k + (\alpha - y))(k - (\alpha - y)) - L \\
&= k^2 - (\alpha - y)^2 - L \\
&= k^2 - \alpha^2 - y^2 + 2\alpha y - L \\
&= k^2 - k^2 + L - y^2 + 2\alpha y - L \\
&= -y^2 + 2\alpha y \\
&= (2\alpha - y)y
\end{aligned}$$

with $y(0) = x(0) - k + \alpha = \alpha$. So by the previous part, $\lim_{t\to\infty}(y) = 2\alpha$

$$\begin{aligned}
\Rightarrow \lim_{t\to\infty} x(t) &= 2\alpha + k - \alpha \\
&= k + \alpha, \text{ as required.}
\end{aligned}$$

STEP II 1995 Q9



$M(X)$: $(l-d)R_y = lR_z$

$\Rightarrow R_y = \frac{l}{l-d}R_z$ or $R_z = \frac{l-d}{l}R_y$

At equilibrium, $F_y = \mu R_y$, $F_z = \mu R_z$

$R(\nearrow)$: $mg\sin\alpha = F_y + F_z$

$= \mu(R_y + R_z)$

$= \mu\left(1 + \frac{l}{l-d}\right)R_z$ or $\mu\left(1 + \frac{l-d}{l}\right)R_y$

$= \mu\frac{2l-d}{l-d}R_z$ or $\mu\frac{2l-d}{l}R_y$

$R(\searrow)$: $mg\cos\alpha + R_z = R_y$

$\Rightarrow mg\cos\alpha = R_y - R_z$

$= \left(\frac{l}{l-d} - 1\right)R_z$ or $\left(1 - \frac{l-d}{l}\right)R_y$

$= \frac{d}{l-d}R_z$ or $\frac{d}{l}R_y$

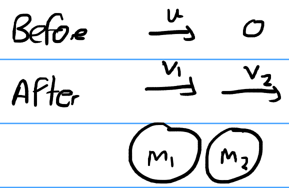
Dividing, $\frac{mg\sin\alpha}{mg\cos\alpha} = \frac{\mu\frac{2l-d}{l-d}}{\frac{d}{l-d}}$ or $\frac{\mu\frac{2l-d}{l}}{\frac{d}{l}}$

$\Rightarrow \tan\alpha = \mu\frac{2l-d}{d}$

$\Rightarrow \mu \geq \frac{d}{2l-d}\tan\alpha$. When $l = 2d$, $\mu \geq \frac{d}{4d-d}\tan\alpha = \frac{1}{3}\tan\alpha$, as required.

STEP II 1995 Q10

For the first collision:



CoM: $m_1 u = m_1 v_1 + m_2 v_2 \quad (1)$

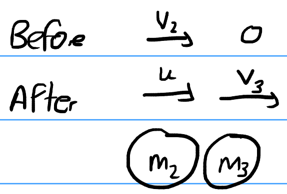
Restitution: $eu = -v_1 + v_2 \quad (2)$

$$+ \quad \underline{m_1 eu = -m_1 v_1 + m_1 v_2} \qquad m_1 \times (2)$$

$$m_1 u(1+e) = v_2(m_1+m_2) \qquad (1) + m_1(2)$$

$$\Rightarrow v_2 = \frac{m_1 u(1+e)}{(m_1+m_2)}$$

For the second collision,



CoM: $m_2 v_2 = m_2 u + m_3 v_3 \quad (1)$

Restitution: $ev_2 = -u + v_3 \quad (2)$

$$\underline{m_3 e v_2 = -m_3 u + m_3 v_3} \qquad m_3 \times (2)$$

$$v_2(m_2 - e m_3) = u(m_2+m_3) \qquad (1) - m_3 \times (2)$$

$$\text{so } u = \frac{v_2(m_2 - e m_3)}{m_2+m_3}$$

$$\Rightarrow \cancel{u} = \cancel{u} \frac{m_1(1+e)(m_2-em_3)}{(m_1+m_2)(m_2+m_3)}$$

$$\Rightarrow m_1 + m_2 = m_1 \frac{(1+e)(m_2-em_3)}{(m_2+m_3)}$$

$$\Rightarrow m_2 = m_1\left(\frac{(1+e)(m_2-em_3)}{(m_2+m_3)} - 1\right)$$

$$= m_1\left(\frac{(1+e)(m_2-em_3)-(m_2+m_3)}{(m_2+m_3)}\right)$$

$$\Rightarrow m_1 = \frac{m_2(m_2+m_3)}{(1+e)(m_2-em_3)-m_2-m_3}$$

We know $m_1 > 0$ (and the numerator is clearly positive), so

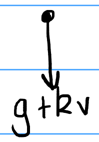
$$(1+e)(m_2-em_3)-m_2-m_3 > 0$$

$$\Rightarrow m_2 + m_2e - em_3 - e^2m_3 \quad - m_2 - m_3 > 0$$

$$\Rightarrow m_2e > m_3(1+e+e^2),$$ as required.

STEP II 1995 Q11

Consider first the particle projected upwards.



$$\frac{d^2x}{dt^2} = -g - kv$$

$$\Rightarrow \frac{d^2x}{dt^2} + k\frac{dx}{dt} = -g$$

C.F.: $m^2 + km = 0 \Rightarrow m = 0$ or $-k$

so $x = A + Be^{-kt}$

P.I $x = \lambda t \Rightarrow k\lambda = -g \Rightarrow \lambda = -g/k$

So $x = A + Be^{-kt} - \frac{g}{k}t$

Now $x(0) = 0 \Rightarrow A + B = 0$

$\dot{x}(u) = 0 \Rightarrow -kB - \frac{g}{k} = u \Rightarrow B = \frac{-u}{k} - \frac{g}{k^2}$, $A = \frac{u}{k} + \frac{g}{k^2}$

So $x = \frac{1}{k}\left[\left(u + \frac{g}{k}\right) - \left(u + \frac{g}{k}\right)e^{-kt} - \frac{gt}{k}\right]$

$\dot{x} = \left(u + \frac{g}{k}\right)e^{-kt} - \frac{g}{k} = 0$ at the top of the trajectory.

$\Rightarrow e^{-kt} = \frac{g}{uk+g}$

$\Rightarrow t = \frac{1}{k}\ln\left|\frac{uk+g}{g}\right| = T.$

At this time, the particle's displacement from its original position is

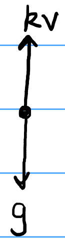
$$\frac{1}{k}\left[\left(u + \frac{g}{k}\right) - \left(u + \frac{g}{k}\right)\cdot\frac{g}{uk+g} - \frac{g}{k}\ln\left(\frac{uk+g}{g}\right)\right]$$

$$= \frac{1}{k}\left[u + \frac{g}{k} - \left(\frac{uk+g}{k}\right)\cdot\frac{g}{uk+g} - \frac{g}{k}\ln\left(\frac{uk+g}{g}\right)\right]$$

$$= \frac{1}{k}\left[u + \frac{g}{k} - \frac{g}{k} - \frac{g}{k}\ln\left(\frac{uk+g}{g}\right)\right]$$

$$= \frac{1}{k}\left(u - \frac{g}{k}\ln\left(\frac{uk+g}{g}\right)\right)$$

Now we consider a particle moving downwards.



$$g - kv = \frac{d^2x}{dt^2}$$
$$\Rightarrow \frac{d^2x}{dt^2} + kv = g$$

The solution is (similar to before) $x = A + Be^{-kt} + \frac{g}{k}t$

$x(0) = 0 \Rightarrow A + B = 0$

$\dot{x}(0) = 0 \Rightarrow -kB + \frac{g}{k} = 0 \Rightarrow B = \frac{g}{k^2},\ A = \frac{-g}{k^2}$

So $x = \frac{-g}{k^2} + \frac{g}{k^2}e^{-kt} + \frac{g}{k}t$

After $T+t$ seconds, the first particle has moved

$$\frac{-g}{k^2} + \frac{g}{k^2}e^{-k(T+t)} + \frac{g}{k}(T+t) \quad \text{downwards from its starting point.}$$

At the same time, the second particle has moved

$$\frac{-g}{k^2} + \frac{g}{k^2}e^{-kt} + \frac{g}{k}t - \frac{1}{k}\left(u - \frac{g}{k}\ln\left(\frac{uk+g}{g}\right)\right) \quad \text{downwards from its starting point.}$$

So the distance between the particles is

$$\cancel{\frac{-g}{k^2}} + \frac{g}{k^2}e^{-k(T+t)} + \frac{g}{k}(T+\cancel{t}) + \cancel{\frac{g}{k^2}} - \frac{g}{k^2}e^{-kt} - \cancel{\frac{g}{k}t} + \frac{1}{k}\left(u - \frac{g}{k}\ln\left(\frac{uk+g}{g}\right)\right) + d$$

$$= \frac{g}{k^2}e^{-kt}\left(e^{-kT} - 1\right) + \frac{g}{k}T + \frac{u}{k} - \frac{g}{k^2}\ln\left(\frac{uk+g}{g}\right) + d$$
$$= \frac{g}{k^2}e^{-kt}\left(\frac{g}{uk+g} - 1\right) + \cancel{\frac{g}{k^2}\ln\left(\frac{uk+g}{g}\right)} + \frac{u}{k} - \cancel{\frac{g}{k^2}\ln\left(\frac{uk+g}{g}\right)} + d$$

$$= d + \frac{u}{k} - \frac{g}{k^2}e^{-kt}\left(\frac{uk}{uk+g}\right)$$

as $t \longrightarrow \infty$, the limit of this is $d + \frac{u}{k}$, as required.

STEP II 1995 Q12

Let p_w = P(Percy wins).

Then $p_w = p + (1-p)(1-r)p_w$

$\Rightarrow p_w(1+p+r-pr-1) = p$

$\Rightarrow p_w = \frac{p}{p+r-pr}$

i) If he hits Cuthbert, then B and A are in the situation from the previous part, with $p = 3/5$, $r = 2/5$. So $P(B \text{ wins}) = \frac{3/5}{3/5+2/5-6/25} = \frac{15}{15+10-6} = \frac{15}{19}$

So $P(C \text{ wins}) = 1 - 15/19 = 4/19$

ii) If he hits Bertie, then Cuthbert hits him with probability 1. So P(C wins = 0).

iii) B hits C with probability $3/5$. Then A and B face off, and A wins with probability $\frac{2/5}{3/5+2/5-6/25} = \frac{10}{19}$. If B misses C with probability $2/5$, then C hits B with probability 1. Then A must hit C (with probability $2/5$) or loses.

So $P(A \text{ wins}) = \frac{3}{5} \times \frac{10}{19} + \frac{2}{5}$

$= \frac{6}{19} + \frac{2}{5}$

$= \frac{226}{475}$.

So Algernon should deliberately miss, and then he wins with probability $\frac{226}{475}$.

STEP II 1995 Q13

Each passenger independently doesn't turn up. We can model the number of passengers who don't show up as $X \sim B(N+k, \varepsilon)$. For large N, and ε small, we can approximate this with a $Po((N+k)\varepsilon)$ distribution. But $k \ll N$, so this is approximately $Po(N\varepsilon)$. So

$$P(r \text{ passengers don't turn up}) = \frac{\lambda^r}{r!}e^{-\lambda} \quad \text{with } \lambda = N\varepsilon.$$

Sales are $A(N+k)$. If $r \geq k$, everyone can get on the plane. Otherwise, we pay $k-r$ passengers compensation.

So profit is $A(N+k) - B\max(0, k-r)$.

$u_0 = AN$
$u_1 = A(N+1) - Be^{-\lambda}$
$u_2 = A(N+2) - B(2e^{-\lambda} + \lambda e^{-\lambda})$
$u_3 = A(N+3) - B(3e^{-\lambda} + 2\lambda e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda})$

$$\text{So } v_k = u_{k+1} - u_k = A - B(e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda} + \text{—} + \frac{\lambda^k}{k!}e^{-\lambda})$$
$$= A - Be^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2!} + \text{—} + \frac{\lambda^k}{k!}e^{-\lambda})$$

This is decreasing as for increasing k, the magnitude of the second term increases. So $v_k > v_{k+1}$.

In the limit as $k \to \infty$, $v_k \to A - Be^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2!} + \dots)$
$$= A - Be^{-\lambda}e^{\lambda}$$
$$= A - B < 0$$

So the v_k are decreasing and their limit is negative. So Fly By Night should pick k maximal so that v_{k-1} is positive but v_k is negative.

STEP II 1995 Q14

$$\int_{-\infty}^{\infty} Ax^2 e^{-x^2/2}\,dx \qquad u \;\; Ax \qquad v' \;\; xe^{-x^2/2}$$
$$u' \;\; A \qquad v \;\; -e^{-x^2/2}$$

$$= \left[-Axe^{-x^2/2}\right]_{-\infty}^{\infty} + A\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

$$= 0 \qquad + A\sqrt{2\pi} = 1$$ as it is a probability distribution.

So $A = \frac{1}{\sqrt{2\pi}}$.

i) $$P(X > 87.3) = \frac{1}{\sqrt{2\pi}}\int_{87.3}^{\infty} x^2 e^{-x^2/2}\,dx$$

$$= \left[\frac{-1}{\sqrt{2\pi}} xe^{-x^2/2}\right]_{87.3}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{87.3}^{\infty} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \times 87.3 \times \exp(-87.3^2/2) + \frac{1}{\sqrt{2\pi}}\int_{87.3}^{\infty} e^{-x^2/2}\,dx$$

Both terms are tiny. The exponential in the first term is tiny and the second term is $P(Z > 87.3)$ where $Z \sim N(0,1)$. So we would certainly reject the hypothesis that $\mu = 0$.

ii) $EX = 0$ so $E\bar{X} = 0$

$$\text{Var}X = EX^2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-x^2/2}\,dx$$
$$= E(Z^4) \text{ where } Z \sim N(0,1)$$
$$= 3.$$

So $\text{Var}\bar{X} = \frac{3}{1000} = 0.003 \Rightarrow sd(\bar{X}) = 0.055$.

So 0.23 is about 4σ from the mean. This is unlikely, but not near-impossible as before. So I would cautiously reject my hypothesis, accepting that I could be wrong.