STEP I 1995 Q1

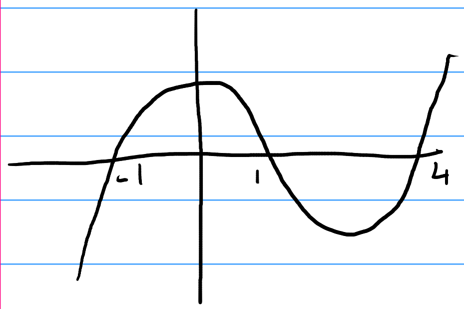
(i) $x^3 - 4x^2 - x + 4 \geq 0$

Note that $4^3 - 4\times 4^2 - 4 + 4 = 0$, so $(x-4)$ is a factor.

$$\begin{array}{r|l} & x^2 + 0x - 1 \\ \hline x-4 & x^3 - 4x^2 - x + 4 \\ & x^3 - 4x^2 \\ \hline & 0 - x \\ & 0 + 0 \\ \hline & -x + 4 \end{array}$$

So $(x-4)(x^2-1) \geq 0$

$\Rightarrow (x-4)(x+1)(x-1) \geq 0$



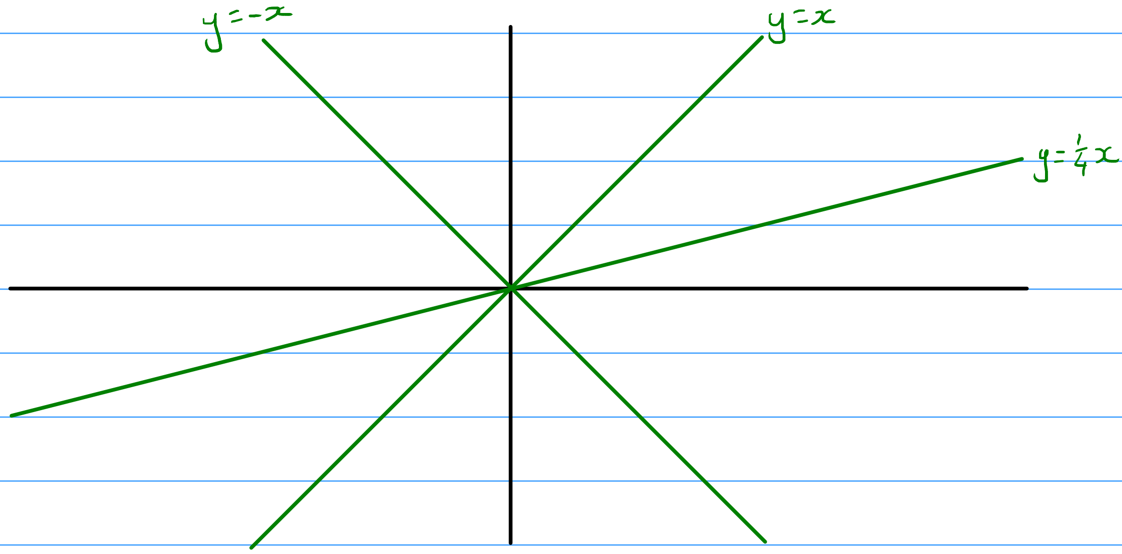

So $-1 \leq x \leq 1$ or $x \geq 4$.

(ii) $x^3 - 4x^2y - xy^2 + 4y^3 = 0$

$\Rightarrow (x-4y)(x+y)(x-y) = 0$ using the factorisation from (i)

So the lines are $y = \frac{1}{4}x$, $y = -x$, $y = x$

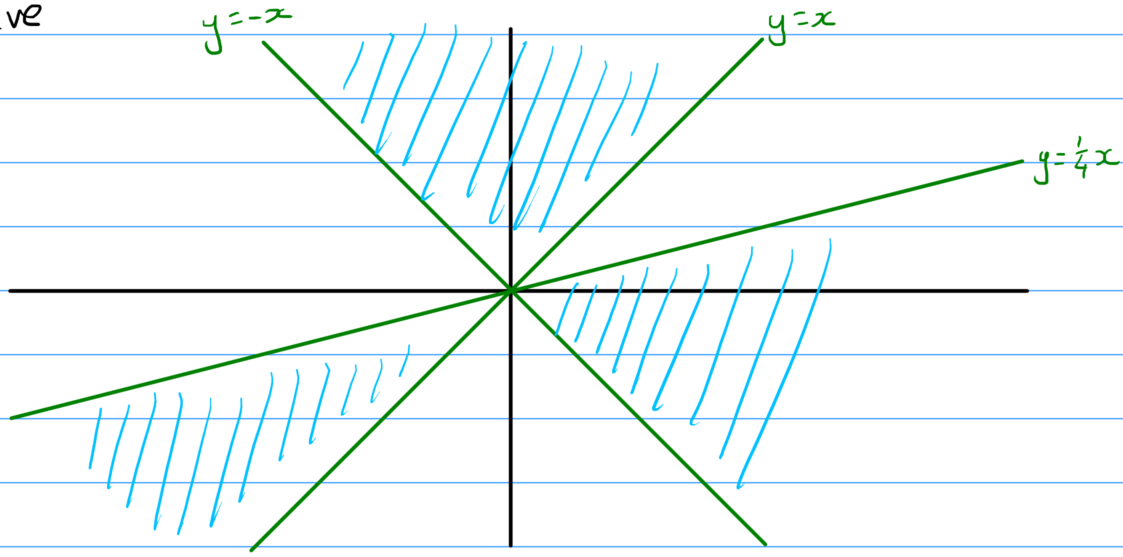
(iii)




Note each time we cross a line, one of the brackets changes sign. So if we can find whether or not one region satisfies the inequality we can then alternate around the plane. Consider $x=1, y=0$. Then

$$1^3 - 4\times1^2\times0 - 1\times0^2 + 4\times0^3 = 1 \geqslant 0,$$

so $(1,0)$ satisfies the inequality. So, shading the regions that satisfy the inequality, we have



STEP I 1995 Q2

(i) $S = \int \frac{\cos x}{\cos x + \sin x}\,dx$, $T = \int \frac{\sin x}{\cos x + \sin x}\,dx$

So $S + T = \int \frac{\cos x + \sin x}{\cos x + \sin x}\,dx$

$= \int dx$

$= x + c$

$S - T = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

$= \ln|\cos x + \sin x| + c$ (cos $\frac{d}{dx}(\cos x + \sin x) = \cos x - \sin x$)

So, $S = \frac{1}{2}x + \frac{1}{2}\ln|\cos x + \sin x| + c$

$T = \frac{1}{2}x - \frac{1}{2}\ln|\cos x + \sin x| + c$

(ii) $\int_{1/4}^{1/2} (1-4x)\sqrt{\frac{1}{x} - 1}\,dx$

$= \int_{1/4}^{1/2} (1-4x)\sqrt{\frac{1-x}{x}}\,dx$ $x = \sin^2 t$, $dx = 2\sin t\cos t\,dt$

$= \int_{\pi/6}^{\pi/4} (1-4\sin^2 t)\cdot\sqrt{\frac{1-\sin^2 t}{\sin^2 t}}\cdot 2\sin t\cos t\,dt$

$= \int_{\pi/6}^{\pi/4} (1-4\sin^2 t)\cdot\frac{\cos t}{\sin t}\cdot 2\sin t\cos t\,dt$

$= \int_{\pi/6}^{\pi/4} 2(1-4\sin^2 t)\cos^2 t\,dt$

$= \int_{\pi/6}^{\pi/4} 2\cos^2 t - 2(2\sin t\cos t)^2\,dt$

$$= \int_{\pi/6}^{\pi/4} 2\cos^2 t - 2\sin^2 2t \, dt$$

$$= \int_{\pi/6}^{\pi/4} \cos 2t + \cancel{1} - \cancel{1} + \cos 4t \, dt$$

$$= \left[\frac{1}{2}\sin 2t + \frac{1}{4}\sin 4t\right]_{\pi/6}^{\pi/4}$$

$$= \left(\frac{1}{2} + 0\right) - \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8}\right)$$

$$= \frac{1}{2} - \frac{3\sqrt{3}}{8}$$

STEP I 1995 Q3

(i) $\sum_{r=1}^{n} f(r) - f(r-1) = \cancel{f(1)} - f(0)$

$+ \cancel{f(2)} - \cancel{f(1)}$

$+ \cancel{f(3)} - \cancel{f(2)}$

$+$

$+ \cancel{f(n-1)} - \cancel{f(n-2)}$

$+ f(n) - \cancel{f(n-1)}$

$= f(n) - f(0).$

(ii) $f(r) - f(r-1) = r^2(r+1)^2 - (r-1)^2 r^2$

$= r^2[(r+1)^2 - (r-1)^2]$

$= r^2[r^2 + 2r + 1 - r^2 + 2r - 1]$

$= 4r^3$

So, $\sum_{r=1}^{n} r^3 = \frac{1}{4}\sum_{r=1}^{n} f(r) - f(r-1)$

$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{4}(1)^2(0)^2$

$= \frac{1}{4}n^2(n+1)^2.$

(iii) $1^3 - 2^3 + 3^3 - 4^3 + \dots + (2n+1)^3$

$= 1^3 + 2^3 + 3^3 + \text{—} + (2n+1)^3 - 2(2^3 + 4^3 + 6^3 + \text{—} + (2n)^3)$

$= \frac{1}{4}(2n+1)^2(2n+2)^2 - 2((2\times1)^3 + (2\times2)^3 + (2\times3)^3 + \text{—} + (2\times n)^3)$

$= \frac{1}{4}(2n+1)^2(2n+2)^2 - 16(1^3 + 2^3 + \text{——} + n^3)$

$= \frac{1}{4}(2n+1)^2(2n+2)^2 - 4n^2(n+1)^2$

$= (2n+1)^2(n+1)^2 - 4n^2(n+1)^2$

$= (n+1)^2[(2n+1)^2 - 4n^2]$

$= (n+1)^2[4n^2 + 4n + 1 - 4n^2]$

$= (4n+1)(n+1)^2$

STEP I 1995 Q4

$$\sin 5\theta = \mathrm{Im}\left[(\cos\theta + i\sin\theta)^5\right]$$
$$= \mathrm{Im}\left[\cos^5\theta + 4i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\right]$$
$$= \cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= \sin\theta\left[\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta\right]$$
$$= \sin\theta\left[\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2\right]$$
$$= \sin\theta\left[\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta\right]$$
$$= \sin\theta\left[16\cos^4\theta - 12\cos^2\theta + 1\right],$$ as required.

Setting $\theta = \pi/5$, and noting $\sin\pi = 0$,

$$0 = \sin\pi/5\left[16\cos^4\pi/5 - 12\cos^2\pi/5 + 1\right]$$

$\sin\pi/5 \neq 0$, so $16\cos^4\pi/5 - 12\cos^2\pi/5 + 1 = 0$

$$\Rightarrow \cos^2\pi/5 = \frac{12 \pm \sqrt{144 - 4\times 16}}{32}$$
$$= \frac{12 \pm \sqrt{80}}{32}$$
$$= \frac{12 \pm 4\sqrt{5}}{32}$$
$$= \frac{3 \pm \sqrt{5}}{8}$$

Now, $\cos^2\pi/5 > \cos^2\pi/4 = \frac{1}{2}$, so we must have $\cos^2\pi/5 = \frac{3+\sqrt{5}}{8}$

Now $\left(\frac{1}{4}(1+\sqrt{5})\right)^2 = \frac{1+5+2\sqrt{5}}{16}$

$= \frac{3+\sqrt{5}}{8}$, and $\cos\pi/5 > 0$, so $\cos\pi/5 = \frac{1}{4}(1+\sqrt{5})$.

STEP I 1995 Q5

$$f(x) = nx - \binom{n}{2}\frac{x^2}{2} + \binom{n}{3}\frac{x^3}{3} + \_\_\_ + (-1)^{r+1}\binom{n}{r}\frac{x^r}{r} + \_\_\_ + (-1)^{n+1}\frac{x^n}{n}$$

$$f'(x) = n - \binom{n}{2}x + \binom{n}{3}x^2 + \_\_\_ + (-1)^{r+1}\binom{n}{r}x^{r-1} + \_\_\_ + (-1)^{n+1}x^{n-1}$$

Note $1-(1-x)^n$

$$= 1 - \left(1 - nx + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \_\_\_ + (-1)^n x^n\right)$$

$$= nx - \binom{n}{2}x^2 + \binom{n}{2}x^3 + \_\_\_ + (-1)^{n+1}x^n$$

$$= xf'(x)$$

So $f'(x) = \dfrac{1-(1-x)^n}{x}$, as required.

So $f(x) = \displaystyle\int_0^x \frac{1-(1-x)^n}{x}$ $\qquad y = 1-x,\ dx = -dy$

$$= -\int_1^{1-x} \frac{1-y^n}{1-y}\,dy$$

$$= \int_{1-x}^{1} \frac{1-y^n}{1-y}\,dy$$

So $f(1) = \displaystyle\int_0^1 \frac{1-y^n}{1-y}\,dy = \int_0^1 \frac{(1-y)(y^{n-1}+y^{n-2}+\_\_\_+1)}{(1-y)}$

$$= \int_0^1 y^{n-1} + y^{n-2} + \_\_\_ + 1$$

$$= \left[\frac{1}{n}y^n + \frac{1}{n-1}y^{n-1} + \_\_\_ y\right]_0^1$$

$$= \frac{1}{n} + \frac{1}{n-1} + \_\_\_ + 1$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \_\_\_ + \frac{1}{n}, \text{ as required.}$$

STEP I 1995 Q6

(i) $\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}=e^{2x}$

$u=1/y$, so $\frac{du}{dx}=\frac{-1}{y^2}\frac{dy}{dx}$

$\Rightarrow -\frac{du}{dx}+u=e^{2x}$

$\Rightarrow \frac{du}{dx}-u=-e^{2x}$

$IF=e^{-\int dx}=e^{-x}$

$\Rightarrow \frac{d}{dx}(ue^{-x})=-e^{x}$

$\Rightarrow ue^{-x}=-e^{x}+A$

$\Rightarrow u=Ae^{x}-e^{2x}$

$\Rightarrow y=\frac{1}{Ae^{x}-e^{2x}}$

(ii) $\frac{1}{y^3}\frac{dy}{dx}+\frac{1}{y^2}=e^{2x}$

Set $u=1/y^2$, so $\frac{du}{dx}=\frac{-2}{y^3}\frac{dy}{dx}$

$\Rightarrow \frac{-1}{2}\frac{du}{dx}+u=e^{2x}$

$\Rightarrow \frac{du}{dx}-2u=-2e^{2x}$

$IF=e^{\int -2dx}=e^{-2x}$, so

$\frac{d}{dx}(ue^{-2x})=-2$

$\Rightarrow ue^{-2x}=-2x+A$

$\Rightarrow u=-2xe^{2x}+Ae^{2x}$

$\Rightarrow y=\frac{1}{\sqrt{Ae^{2x}-2xe^{2x}}}$

# STEP I 1995 Q7

Every point on the perpendicular bisector of A and B is equidistant from A and B. Similarly, every point on the perpendicular bisector of B and C is equidistant from B and C. Because the points are non-collinear, these lines are not parallel and so meet at a point that is equidistant from all three points

i) We have $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$, $\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, and $\overrightarrow{ON} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.

$$\begin{aligned}\overrightarrow{AQ} &= \overrightarrow{AO} + \overrightarrow{OQ}\\ &= -\overrightarrow{OA} + \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\\ &= \overrightarrow{OB} + \overrightarrow{OC}\end{aligned}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$\begin{aligned}\text{So } \overrightarrow{AQ} \cdot \overrightarrow{BC} &= (\overrightarrow{OB} + \overrightarrow{OC}) \cdot (\overrightarrow{OC} - \overrightarrow{OB})\\ &= |\overrightarrow{OC}|^2 - |\overrightarrow{OB}|^2\\ &= 0 \text{ as } |\overrightarrow{OB}| = |\overrightarrow{OC}|\end{aligned}$$

So $\overrightarrow{AQ}$ and $\overrightarrow{BC}$ are perpendicular.

ii) The midpoint of AQ is $\overrightarrow{OA} + \frac{1}{2}(\overrightarrow{AQ})$

$$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}$$

$$\begin{aligned}\text{So } \overrightarrow{N\,AQ_{mid}} &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC} - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})\\ &= \frac{1}{2}\overrightarrow{OA}.\end{aligned}$$

Similarly, the results for BQ and CQ are $\frac{1}{2}\overrightarrow{OB}$ and $\frac{1}{2}\overrightarrow{OC}$ respectively. These are all the same length as $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}|$.

Midpoint of AB is $\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$$= \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$$

So $\overrightarrow{AB_{mid}N} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA} - \frac{1}{2}\overrightarrow{OB}$

$= \frac{1}{2}\overrightarrow{OC}$.

Similar results follow for BC and AC, and again these all have magnitude $\frac{1}{2}|\overrightarrow{OA}|$.

So, all six of these points are equidistant from N.

STEP I 1995 Q8

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = h(x)$$

Setting $y=1$, we get $g(x) = h(x)$.

Setting $y=x$, $f(x) + xg(x) = h(x) \Rightarrow f(x) + xg(x) = g(x)$

$$\Rightarrow f(x) = (1-x)g(x)$$

Setting $y = x^{-1}$, $\frac{2}{x^3} - \frac{1}{x^2}f(x) + \frac{1}{x}g(x) = h(x)$

$$\Rightarrow \frac{2}{x^3} - \frac{1}{x^2}\left(g(x) - xg(x)\right) + \frac{1}{x}g(x) = g(x)$$

$$\Rightarrow \frac{2}{x^3} - \frac{1}{x^2}g(x) + \frac{1}{x}g(x) + \frac{1}{x}g(x) = g(x)$$

$$\Rightarrow \frac{2}{x^3} = g(x)\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)$$

$$= g(x)\left(\frac{x^2-2x+1}{x^2}\right)$$

$$\Rightarrow g(x) = \frac{2}{x(x^2-2x+1)}$$

$$= \frac{2}{x(x-1)^2}$$

So $h(x) = \frac{2}{x(x-1)^2}$

$$f(x) = \frac{2}{x(x-1)^2} - \frac{2}{(x-1)^2}$$

$$= \frac{2(1-x)}{x(x-1)^2}$$

$$= \frac{2}{x(1-x)}$$

So, $\frac{d^2y}{dx^2} + \frac{2}{x(1-x)}\frac{dy}{dx} + \frac{2}{x(1-x)^2}y = \frac{2}{x(1-x)^2}$

Substituting $y = ax + b + \frac{c}{x}$

$$\frac{dy}{dx} = a - \frac{c}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2c}{x^3}$$

So $\frac{2c}{x^3} + \frac{2}{x(1-x)}\left(a - \frac{c}{x^2}\right) + \frac{2}{x(1-x)^2}\left(ax + b + \frac{c}{x}\right) = \frac{2}{x(1-x)^2}$

$\Rightarrow c + \frac{1}{1-x}(ax^2 - c) + \frac{1}{(1-x)^2}(ax^3 + bx^2 + cx) = \frac{x^2}{(1-x)^2}$

$\Rightarrow c(1-x)^2 + (1-x)(ax^2 - c) + (ax^3 + bx^2 + cx) = x^2$

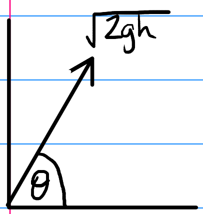
$\Rightarrow c(1 - 2x + x^2) + (ax^2 - c - ax^3 + cx) + (ax^3 + bx^2 + cx) = x^2$

$\Rightarrow x^3(-a+a) + x^2(a+b+c) + x(-2c+c+c) + (c-c) = x^2$

$\Rightarrow a + b + c = 1.$

STEP I 1995 Q9

We shall find, for a fixed horizontal distance $x$ from the origin, the maximum possible height of a trajectory at that point.



Horizontally, $x = \sqrt{2gh}\cos\theta t$

$$\Rightarrow t = \frac{x}{\sqrt{2gh}\cos\theta} \quad (*)$$

Vertically, $s = ut + \frac{1}{2}at^2$

$$\Rightarrow y = \sqrt{2gh}\sin\theta t - \frac{1}{2}gt^2$$

Substituting in $(*)$, $y = x\tan\theta - \frac{1}{2}g\frac{x^2}{2gh\cos^2\theta}$

$$= x\tan\theta - \frac{x^2}{4h\cos^2\theta}$$

$$= x\tan\theta - \frac{x^2}{4h}\sec^2\theta \quad (\dagger)$$

taking $x$ fixed, and differentiating w.r.t. $\theta$,

$$\frac{dy}{d\theta} = x\cancel{\sec^2\theta} - \frac{x^2}{4h}2\cancel{\sec^2\theta}\tan\theta = 0$$

$$\Rightarrow 1 - \frac{x}{2h}\tan\theta = 0$$

$$\Rightarrow \tan\theta = \frac{2h}{x}$$

$$\Rightarrow \sec^2\theta = 1 + \frac{4h^2}{x^2}$$

Substituting into $(\dagger)$,

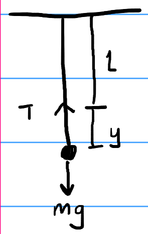
$$y = x.\frac{2h}{x} - \frac{x^2}{4h}\left(1 + \frac{4h^2}{x^2}\right)$$

$$= 2h - \frac{x^2}{4h} - h$$

$$\Rightarrow x^2 = 4h^2 - 4hy$$

$$= 4h(h-y), \text{ as required.}$$

STEP I 1995 Q10




In equilibrium, $T = mg \Rightarrow \frac{\lambda y}{l} = mg$

$\Rightarrow y = \frac{mgl}{\lambda}$

So total length is $l + \frac{mgl}{\lambda}$

$= l\left(1 + \frac{mg}{\lambda}\right)$

$F = ma \Rightarrow mg - \frac{\lambda y}{l} = m\ddot{x}$

Note $x = y - \frac{mlg}{\lambda}$, so

$$mg - \frac{\lambda}{l}x - mg = m\ddot{x}$$

$$\Rightarrow \ddot{x} = \frac{-\lambda}{lm}x$$

$$\Rightarrow x = A\cos\sqrt{\frac{\lambda}{lm}}t + B\sin\sqrt{\frac{\lambda}{lm}}t$$

But $x(0) = 0 \Rightarrow x = B\sin\sqrt{\frac{\lambda}{lm}}t$

$\dot{x}(0) = u_0 \Rightarrow B = u_0\sqrt{\frac{ml}{\lambda}}$, so $x = u_0\sqrt{\frac{ml}{\lambda}}\sin\sqrt{\frac{\lambda}{lm}}t$

Note that, for SHM, $v^2 = \omega^2(a^2 - x^2)$

$a = \text{amplitude} = u_0\sqrt{\frac{ml}{\lambda}}$

$\omega = \sqrt{\frac{\lambda}{ml}}$, so $v^2 = \frac{\lambda}{ml}\left(u_0^2\frac{ml}{\lambda} - x^2\right)$

$$\Rightarrow v^2 = u_0^2 - \frac{\lambda}{ml}x^2$$

$$\Rightarrow v^2 + \frac{\lambda}{lm}x^2 = u_0^2,$$ as required.

At distance $h$, $v^2 = u_0^2 - \frac{\lambda}{ml}h^2$, so rebound speed$^2$ is $e^2\left(u_0^2 - \frac{\lambda}{ml}h^2\right)$.

We need to find the new amplitude. Using $v^2 = \omega^2(a^2 - x^2)$,

$$e^2\left(u_0^2 - \frac{\lambda}{ml}h^2\right) = \frac{\lambda}{ml}(a^2 - h^2)$$

$$\Rightarrow a^2 = \frac{e^2 ml}{\lambda}\left(u_0^2 - \frac{\lambda}{ml}h^2\right) + h^2$$

The velocity$^2$ at $x=0$ is $u_1^2 = \omega^2 a^2$

$$= \frac{\lambda}{m\ell} \cdot \frac{e^2 m\ell}{\lambda}\left(u_0^2 - \frac{\lambda}{m\ell}h^2\right) + \frac{\lambda}{m\ell}h^2$$

$$= e^2 u_0^2 - e^2\frac{\lambda}{m\ell}h^2 + \frac{\lambda}{m\ell}h^2$$

$$= e^2 u_0^2 + \frac{\lambda}{\ell m}h^2(1-e^2), \text{ as required.}$$

The speed$^2$ when reaching the surface is $e^2(u_0^2 - \frac{\lambda}{m\ell}h^2)$, so the rebound speed$^2$ is $e^4(u_0^2 - \frac{\lambda}{m\ell}h^2)$.

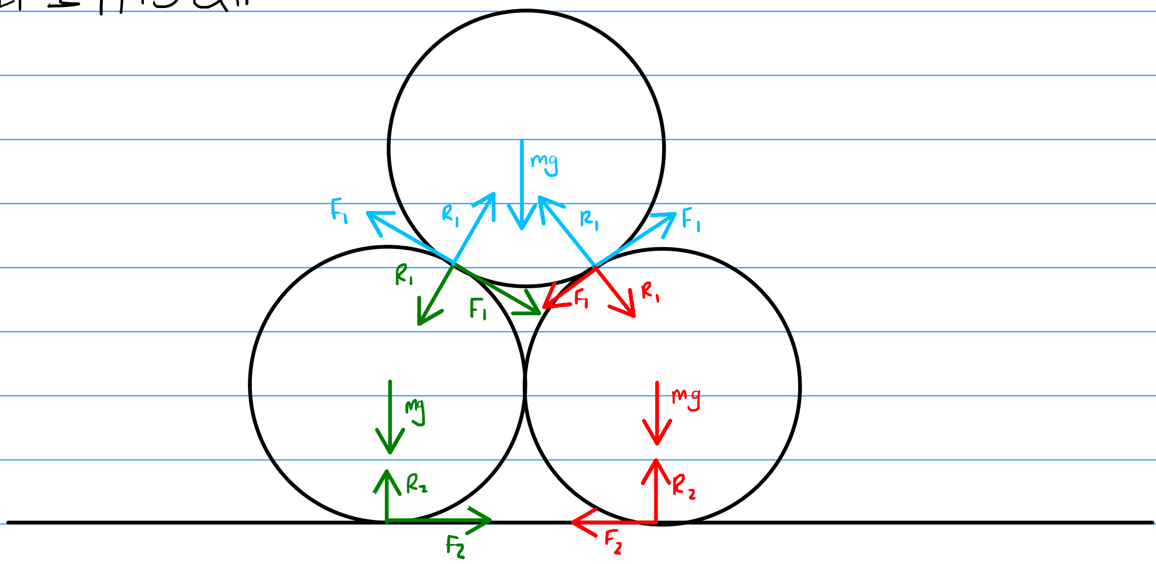
Using the same argument as before, the new amplitude is

$$a^2 = \frac{e^4 m\ell}{\lambda}\left(u_0^2 - \frac{\lambda}{m\ell}h^2\right) + h^2,$$

So $u_2^2 = \frac{\lambda}{m\ell}\left[\frac{e^4 m\ell}{\lambda}\left(u_0^2 - \frac{\lambda}{m\ell}h^2\right) + h^2\right]$

$$= e^4 u_0^2 + \frac{\lambda}{\ell m}h^2(1-e^4), \text{ as required.}$$

STEP I 1995 Q11




Resolving vertically for the whole system,

$$2R_2 = 3mg \Rightarrow R_2 = \tfrac{3}{2}mg$$

Considering the bottom left sphere,

Moments about the centre $\Rightarrow F_1 = F_2$

Resolving horizontally $F_2 + F_1\cos 30^\circ = R_1\cos 60^\circ$

$$\Rightarrow F_2 + F_1\tfrac{\sqrt{3}}{2} = \tfrac{1}{2}R_1$$

$$\Rightarrow (2+\sqrt{3})F_1 = R_1$$

$$\Rightarrow F_1 = \frac{1}{2+\sqrt{3}}R_1$$

$$= (2-\sqrt{3})R_1 \quad \text{, so } \mu \geqslant 2-\sqrt{3} \quad (*)$$

Resolving vertically $R_2 = mg + R_1\cos 30^\circ + F_1\cos 60^\circ$

$$R_2 = mg + \tfrac{\sqrt{3}}{2}R_1 + \tfrac{1}{2}F_1$$

so $\tfrac{3}{2}mg = mg + \tfrac{\sqrt{3}}{2}R_1 + \tfrac{1}{2}F_1$

$$\Rightarrow mg = \sqrt{3}R_1 + F_1$$

$$= \sqrt{3}R_1 + (2-\sqrt{3})R_1$$

$$= 2R_1$$

$$\Rightarrow R_1 = \tfrac{1}{2}mg \Rightarrow F_2 = F_1 = (2-\sqrt{3})\cdot\tfrac{1}{2}mg$$

$$= \tfrac{1}{3}(2-\sqrt{3})R_2 \quad \text{, so } \mu \geqslant \tfrac{1}{3}(2-\sqrt{3}) \quad (+)$$

So for the system to be stable, $(*)$ and $(+)$ are both true, so $\mu \geqslant 2-\sqrt{3}$, as required.

STEP I 1995 Q12

i) $\frac{r}{n}\times\frac{(r-1)}{(n-1)} = \frac{r(r-1)}{n(n-1)}$

ii) There are $(n-r+1)$ spaces for the the group of $r$ players, $r!$ ways of arranging the hockey players, and $(n-r)!$ ways of arranging the non-hockey players, so the probability is

$$\frac{r!(n-r)!(n-r+1)}{n!}$$

iii) There are $(n-r+1)$ gaps. So if $r > n-r+1$

$\Rightarrow r > \frac{n+1}{2}$, the probability is 0.

Else, there are $\binom{n-r+1}{r}$ ways of choosing the gaps, each of which has $r!$ arrangements of the hockey players, and $(n-r)!$ ways of arranging the non-hockey players.

So the probability is

$$\frac{\binom{n-r+1}{r}r!(n-r)!}{n!}$$

$$= \frac{(n-r+1)!r!(n-r)!}{r!n!(n-2r+1)!}$$

$$= \frac{(n-r+1)!(n-r)!}{n!(n-2r+1)!}$$

STEP I 1995 Q13

i) $P(\text{spots } k \text{ cells}) = \sum_{n=k}^{\infty} P(n \text{ cells})\, P(\text{spots } k \text{ from } n)$

$$= \sum_{n=k}^{\infty} \frac{\mu^n e^{-\mu}}{n!} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= e^{-\mu} \left(\frac{p}{1-p}\right)^k \sum_{n=k}^{\infty} \frac{\mu^n}{\cancel{n!}} \cdot \frac{\cancel{n!}}{k!(n-k)!} (1-p)^n$$

$$= e^{-\mu} \left(\frac{p}{1-p}\right)^k \cdot \frac{1}{k!} \sum_{n=k}^{\infty} \frac{[\mu(1-p)]^n}{(n-k)!}$$

$$= e^{-\mu} \left(\frac{p}{1-p}\right)^k \cdot \frac{1}{k!} \cdot \mu^k (1-p)^k \cdot \sum_{n=0}^{\infty} \frac{[\mu(1-p)]^n}{n!}$$

$$= \cancel{e^{-\mu}} \left(\frac{p}{\cancel{1-p}}\right)^k \cdot \frac{1}{k!} \cdot \mu^k \cancel{(1-p)^k}\, \cancel{e^{\mu}}\, e^{-\mu p}$$

$$= \frac{e^{-\mu p} (\mu p)^k}{k!}$$ , which is a $Po(\mu p)$, as required.

ii) $P(\text{second cell is on slide } k) = P(\text{first \& second on } k) + P(\text{first} < k, \text{ second on } k)$

$= P(\text{none in first } k-1)\, P(\geq 2 \text{ on } k) + P(\text{exactly one in first } k-1)\, P(\geq 1 \text{ on } k)$

$= (e^{-\mu p})^{k-1} \cdot (1 - e^{-\mu p} - \mu p e^{-\mu p}) + (k-1)(e^{-\mu p})^{k-2} \mu p e^{-\mu p} (1 - e^{-\mu p})$

$= e^{-\mu p(k-1)} \left(1 - e^{-\mu p} - \mu p e^{-\mu p} + \mu p(k-1)(1 - e^{-\mu p})\right)$

$= e^{-\mu p(k-1)} \left((1 + k\mu p - \mu p)(1 - e^{-\mu p}) + \mu p - \mu p e^{-\mu p} - \mu p\right)$

$= e^{-\mu p(k-1)} \left((1 + k\mu p - \mu p)(1 - e^{-\mu p}) + \mu p(1 - e^{-\mu p}) - \mu p\right)$

$= e^{-\mu p(k-1)} \left((1 + k\mu p)(1 - e^{-\mu p}) - \mu p\right)$, as required.

# STEP I 1995 Q14

i) The maximum of $\sqrt{p(1-p)}$ occurs at the same value of $p$ as the maximum of $p(1-p)$, as the square function is monotone increasing.

$$\frac{d}{dp}(p(1-p)) = 1-2p = 0$$
$$\Rightarrow p = 1/2$$
$$\frac{d^2}{dp^2}(p(1-p)) = -2 < 0 \quad \text{so maximum.}$$

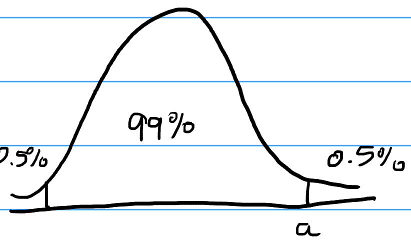
When $p = \frac{1}{2}$, $\sqrt{p(1-p)} = \sqrt{\frac{1}{2} \times \frac{1}{2}} = \frac{1}{2}$.

ii) Use a normal approximation, so the number of women in the sample is $X \sim N(np, np(1-p))$

Then $\hat{p} = \frac{X}{n} \sim N\left(p, \frac{p(1-p)}{n}\right)$

So $\dfrac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$

we want to be right 99% of the time




From tables (or calculator), $a = 2.5758$

so $2.5758 \cdot \sqrt{\frac{p(1-p)}{n}} = 0.01$

$$\Rightarrow n = \left(\frac{2.5758}{0.01}\right)^2 p(1-p)$$

$p \approx \frac{1}{2}$ (and from above $p = \frac{1}{2}$ is the worst case),

So $n \geq 16,587$

(iii) For left-handers, $p \approx 0.1$, so $p(1-p) = 0.09$, so sample size is $\sim 2.5$ times smaller.
For millionaires, $p \approx 0.01$, so $p(1-p) = 0.009$, so sample size is $\sim 25$ times smaller.