STEP III 1994 Q1

$$\int_0^x \operatorname{sech} t \, dt = \int_0^x \frac{1}{\cosh t} dt \qquad \text{Set } u = \sinh t, \text{ so } dt = \frac{du}{\cosh t}$$

$$= \int_0^{\sinh x} \frac{du}{\cosh^2 t}$$

$$= \int_0^{\sinh x} \frac{du}{1+u^2}$$

$$= \Big[\arctan u\Big]_0^{\sinh x}$$

$$= \arctan(\sinh x)$$

Now $I_n = \int_0^x \operatorname{sech}^n t \, dt$. Proceed via integration by parts.

$$= \int_0^x \operatorname{sech}^2 t \operatorname{sech}^{n-2} t$$

$u \quad \operatorname{sech}^{n-2} t \qquad v' \quad \operatorname{sech}^2 t$

$u' \quad (n-2)\operatorname{sech}^{n-3} t \operatorname{sech} t \tanh t \qquad v \quad \tanh t$

$$= \Big[\tanh t \operatorname{sech}^{n-2} t\Big]_0^x - (n-2)\int \operatorname{sech}^{n-2} t \tanh^2 t \, dt$$

$$= \tanh x \operatorname{sech}^{n-2} x - (n-2)\int \operatorname{sech}^{n-2} t - \operatorname{sech}^n t \, dt \quad (\text{as } \tanh^2 t = 1 - \operatorname{sech}^2 t)$$

$$= \tanh x \operatorname{sech}^{n-2} x - (n-2) I_{n-2} - (n-2) I_n$$

So $(n-1) I_n = \tanh x \operatorname{sech}^{n-2} x + (n-2) I_{n-2}$

$$\Rightarrow I_n = \frac{1}{n-1} \tanh x \operatorname{sech}^{n-2} x + \frac{n-2}{n-1} I_{n-2}.$$

So, $I_5 = \frac{1}{4}\tanh x \operatorname{sech}^3 x + \frac{3}{4} I_3$

$= \frac{1}{4}\tanh x \operatorname{sech}^3 x + \frac{3}{4}\left(\frac{1}{2}\tanh x \operatorname{sech} x + \frac{1}{2} I_1\right)$

$= \frac{1}{4}\tanh x \operatorname{sech}^3 x + \frac{3}{8}\tanh x \operatorname{sech} x + \frac{3}{8}\arctan(\sinh x)$

$I_6 = \frac{1}{5}\tanh x \operatorname{sech}^4 x + \frac{4}{5} I_4$

$= \frac{1}{5}\tanh x \operatorname{sech}^4 x + \frac{4}{5}\left(\frac{1}{3}\tanh x \operatorname{sech}^2 x + \frac{2}{3} I_2\right)$

$= \frac{1}{5}\tanh x \operatorname{sech}^4 x + \frac{4}{15}\tanh x \operatorname{sech}^2 x + \frac{8}{15}\operatorname{sech}^2 x$

STEP III 1994 Q2

i) $\left(x^2+\frac{1}{x^2}\right)+10\left(x+\frac{1}{x}\right)+26=0$

$\left(x+\frac{1}{x}\right)^2-2+10\left(x+\frac{1}{x}\right)+26=0$

$(x+1/x)^2+10(x+1/x)+24=0$

Setting $y=x+1/x$, this becomes

$y^2+10y+24=0$

$\Rightarrow (y+6)(y+4)=0$

$\Rightarrow y=-6$ or $y=-4$

$x+\frac{1}{x}=-6$ $\qquad$ $x+\frac{1}{x}=-4$

$x^2-6x+1=0$ $\qquad$ $x^2-4x+1=0$

$x=\frac{-6\pm\sqrt{36-4}}{2}$ $\qquad$ $x=\frac{-4\pm\sqrt{16-4}}{2}$

$=-3\pm2\sqrt{2}$ $\qquad$ $=-2\pm\sqrt{3}$

ii) $x^2+16/x^2+x-4/x-10=0$

$(x-4/x)^2+8+(x-4/x)-10=0$

Setting $y=x-4/x$,

$y^2+y-2=0$

$\Rightarrow (y+2)(y-1)=0$

$\Rightarrow y=1$ or $y=-2$

$x-4/x=1$ $\qquad$ $x-4/x=-2$

$x^2-x-4=0$ $\qquad$ $x^2+2x-4=0$

$x=\frac{1\pm\sqrt{1+16}}{2}$ $\qquad$ $x=\frac{-2\pm\sqrt{4+16}}{2}$

$=\frac{1\pm\sqrt{17}}{2}$ $\qquad$ $=-1\pm\sqrt{5}$

STEP III 1994 Q 3

The intersection of a plane and a sphere is either a point (when the plane lies tangent to the sphere), or a circle.

Start with the intersection of P_1 and P_2.

$3x - y - 1 = 0$ and $x - y + 1 = 0$

$\Rightarrow y = 3x - 1$, so $x - (3x-1) + 1 = 0$

$\Rightarrow -2x + 2 = 0$

$\Rightarrow x = 1, y = 2.$

So the intersection is the line $x=1, y=2, z \in \mathbb{R}$, or a vertical line through $(1,2,0)$.

Now find the intersections of L and S_1, and L and S_2.

L and S_1: $1^2 + 2^2 + z^2 = 7$

$\Rightarrow z = \pm\sqrt{2}$ so $(1,2,\sqrt{2})$ or $(1,2,-\sqrt{2})$

L and S_2: $x^2 + (y-3)^2 - 9 + (z-2)^2 - 4 + 10 = 3$

$\Rightarrow x^2 + (y-3)^2 + (z-2)^2 = 3$

so $1 + 1 + (z-2)^2 = 3$

$\Rightarrow (z-2)^2 = 1$

$\Rightarrow z - 2 = \pm 1$, so $z =$ $(1,2,1)$ or $(1,2,3)$

The order of the points on L is $(1,2,-\sqrt{2})$, $(1,2,1)$, $(1,2,\sqrt{2})$, $(1,2,3)$, in the order $C_1 C_2 C_1 C_2$, so the circles are linked.

STEP III 1994 Q4

$$\left(\frac{dy}{dx}\right)^2 = 4y \Rightarrow \frac{dy}{dx} = \pm 2y^{1/2}$$

$$\Rightarrow \pm\int y^{-1/2}\,dy = \int 2\,dx$$

$$\Rightarrow \pm 2y^{1/2} = 2x + c$$

so $y_1 = (x+c)^2$ or $y_2 = (-x+c)^2$

(Note c is an arbitrary constant so we can replace it with $2c$, $-c$ etc.)

These pass through (a, b^2) and so we obtain

$$y_1 = (x+b-a)^2 \text{ and } y_2 = (-x+a+b)^2$$

Now each solution passes through $(a_i, 1)$ and also the origin.

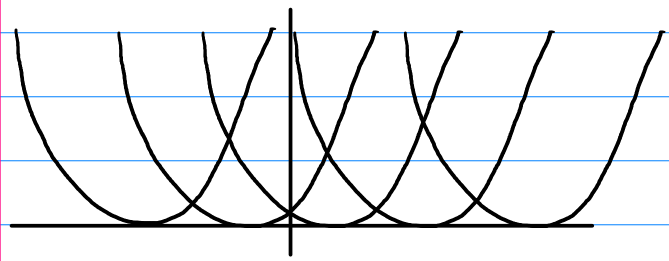
For y_1, we obtain

$0 = (b-a)^2 \Rightarrow a = b$, and so $y_1 = x^2$, which passes through $(1, 1)$, so $a_1 = 1$.

For y_2, we obtain

$0 = (a+b)^2 \Rightarrow a = -b$, and so $y_2 = (-x)^2 = x^2$, which passes through $(-1, 1)$, so $a_2 = 1$.

Note all solutions are in the form $(x+k)^2$ for $k \in \mathbb{R}$, so solutions look like



The common tangent is $y=0$, which is also a solution of the original differential equation.

STEP III 1994 Q5

$f(x) = \arcsin x$

$f'(x) = (1-x^2)^{-1/2}$

$f''(x) = (-1/2)(-2x)(1-x^2)^{-3/2}$

$= x(1-x^2)^{-3/2}$

So $(1-x^2)f''(x) - xf'(x)$ $(*)$

$= (1-x^2)x(1-x^2)^{-3/2} - x(1-x^2)^{-1/2}$

$= x(1-x^2)^{-1/2} - x(1-x^2)^{-1/2}$

$= 0$, as required.

Now want to show that

$$(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) - n^2f^{(n)}(x) = 0$$

Proceed via induction.

For $n=1$, differentiate $(*)$. We obtain

$$(1-x^2)f'''(x) - 2xf''(x) - xf''(x) - f'(x) = 0$$

$\Rightarrow (1-x^2)f^{(3)}(x) - 3xf^{(2)}(x) - f^{(1)}(x) = 0$, so true for $n=1$.

Assume true for $n=k$, so

$$(1-x^2)f^{(k+2)}(x) - (2k+1)xf^{(k+1)}(x) - k^2f^{(k)}(x) = 0$$

Differentiating,

$$(1-x^2)f^{(k+3)}(x) - 2xf^{(k+2)}(x) - (2k+1)f^{(k+1)}(x) - (2k+1)xf^{(k+2)}(x)$$
$$- k^2f^{(k+1)}(x)$$

$$=(1-x^2)f^{(k+3)}(x) - x(2k+1+2)f^{(k+2)}(x) - (k^2+2k+1)f^{(k+1)}(x)$$

$$=(1-x^2)f^{(k+3)}(x) - (2k+3)xf^{(k+2)}(x) - (k+1)^2 f^{(k+1)}(x),$$ so true for $n=k+1$.

True for $n=1$, and if true for $n=k$, then true for $n=k+1$, so true $\forall n>0$.

Now we want to find a Maclaurin series. Setting $x=0$ into the equation, we obtain

$$f^{(n+2)}(0) = n^2 f^{(n)}(0).$$

Also, $f(0) = \arcsin(0) = 0$

$$f^{(1)}(0) = \frac{1}{\sqrt{1-0^2}} = 1$$

So, all even terms are zero. Maclaurin series is

$$f(x) = x + \frac{x^3}{3!} + \frac{3^2 \times x^5}{5!} + \frac{3^2\times 5^2}{7!}x^7 + \dots.$$

$$g(x) = \ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$$

$$=\frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) - \frac{1}{2}\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\right)$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

The coefficient of x^{2n+1} in f is $\dfrac{1^2\times 3^2 \times 5^2 \times \dots \times (2n-1)^2}{(2n+1)!}$

The coefficient of x^{2n+1} in g is $\dfrac{1}{2n+1}$

Then $\dfrac{1}{2n+1} - \dfrac{1^2\times 3^2\times 5^2\times\dots\times(2n-1)^2}{(2n+1)!}$

$$= \frac{1}{2n+1}\left[(2n)! - 1^2\times 3^2 \times 5^2 \times \dots \times (2n-1)^2\right] \qquad (\dagger)$$

And clearly $1^2 \times 3^2 \times \dots \times (2n-1)^2$

$$< (1\times 2)\times(3\times 4)\times \dots \times(2n-1)\times(2n)$$

$$=(2n)!$$

Thus (†) is positive and so the coefficient of x^{2n+1} is greater in the expansion of g than in the expansion of f.

STEP III Q6 1994

Note rotating $\pi/2$ about a is equivalent to translating by $-a$, rotating by $\pi/2$ about the origin, and then translating by a.

Thus $z_2 = i(z_1 - a) + a$

$$z_3 = i(z_2 - b) + b$$
$$= i(i(z_1 - a) + a - b) + b$$
$$= -(z_1 - a) + i(a - b) + b$$
$$z_4 = i(z_3 - c) + c$$
$$= i(-(z_1 - a) + i(a-b) + b - c) + c$$
$$= -i(z_1 - a) - (a - b) + i(b - c) + c$$
$$z_5 = i(z_4 - d) + d$$
$$= i(-i(z_1 - a) - (a - b) + i(b - c) + c - d) + d$$
$$= (z_1 - a) - i(a - b) - (b - c) + i(c - d) + d$$
$$= z_1 - a - b + c + d + i(-a + b + c - d)$$
$$= z_1 + (1 + i)(c - a) + (1 - i)(d - b)$$

Thus for $z_1 = z_5$ we require $(1+i)(c-a) = (1-i)(b-d)$

$$\Rightarrow c - a = \frac{1-i}{1+i}(b-d)$$

But $\frac{1-i}{1+i}$

$$= \frac{1-i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{-2i}{2}$$
$$= -i$$

So $c - a = -i(b - d)$

$\Rightarrow a - c = i(b - d)$.

Geometrically the lines AC and BD are the same length, and perpendicular to each other. Thus ABCD is a square.

Now note rotating by θ is equivalent to multiplying by $e^{i\theta}$. So,

$$z_2 = e^{i\theta}(z_1 - a) + a$$
$$z_3 = e^{i\theta}(e^{i\theta}(z_1 - a) + a - b) + b$$
$$= e^{2i\theta}(z_1 - a) + e^{i\theta}(a - b) + b$$
$$z_4 = e^{i\theta}(e^{2i\theta}(z_1 - a) + e^{i\theta}(a - b) + b - c) + c$$
$$= e^{3i\theta}(z_1 - a) + e^{2i\theta}(a - b) + e^{i\theta}(b - c) + c$$

We want z_4 to be equal to z_1. Note that (for fixed θ, a, b, c), the only non-constant term of z_4 is $e^{3i\theta} z_1$. So we must have $e^{3i\theta} = 1 \Rightarrow \theta = 2\pi/3$ or $4\pi/3$

$$z_4 = z_1 + (1 - e^{i\theta})(e^{2i\theta} a + e^{i\theta} b + c)$$

So $ae^{2i\theta} + be^{i\theta} + c = 0$ (as $e^{i\theta} \neq 1$)

Note also that as a, b, and c are on the unit circle, we can write $a = e^{i\alpha}$, $b = e^{i\beta}$, $c = e^{i\gamma}$.

$$\Rightarrow e^{i(2\theta + \alpha)} + e^{i(\theta + \beta)} + e^{i\gamma} = 0$$
$$\Rightarrow e^{i(2\theta + \alpha - \gamma)} + e^{i(\theta + \beta - \gamma)} + 1 = 0$$

The imaginary part of the first two terms must cancel, so $\mathrm{Im}(e^{i(2\theta + \alpha - \gamma)}) = \mathrm{Im}(e^{i(\theta + \beta - \gamma)})$ and hence $\mathrm{Re}(e^{i(2\theta + \alpha - \gamma)}) = \mathrm{Re}(e^{i(\theta + \beta - \gamma)}) = -1/2$.

Set $u = 2\theta + \alpha - \gamma$ and $-u = \theta + \beta - \gamma$.

Now $\cos u = -1/2 \Rightarrow u = 2\pi/3$ or $4\pi/3 = \theta$ or $-\theta$ (mod 2π)

But if $u = -\theta$ (mod 2π), then $\alpha = \gamma$, contradicting that A, B, and C are distinct. So, $u = \theta$.

Hence $2\theta + \alpha - \gamma = \theta$ $\quad$ Similarly $\theta + \beta - \gamma = -\theta$

$\Rightarrow \alpha = \gamma + \theta$. $\quad$ $\Rightarrow \beta = \gamma + 2\theta$

And as $\theta = 2\pi/3$ or $4\pi/3$, A, B, and C, are equally spaced around the unit circle and hence form an equilateral triangle.

STEP III 1994 Q7

The elements of S_3, with their orders, are

$(1)(2)(3)$	1
$(1)(2\ 3)$	2
$(1\ 3)(2)$	2
$(1\ 2)(3)$	2
$(1\ 2\ 3)$	3
$(1\ 3\ 2)$	3

The elements of $\mathbb{Z}_6$ are, with their orders,

0	1
1	6
2	3
3	2
4	3
5	6

So S_3 and $\mathbb{Z}_6$ are not isomorphic – $\mathbb{Z}_6$ has an element of order 6, but S_3 does not.

Now C_6 and $\mathbb{Z}_6$. We claim $f: C_6 \to \mathbb{Z}_6$ defined by $f(e^{\frac{i\pi n}{3}}) = n$ is a homomorphism.

Proof: $e^{\frac{i\pi n}{3}} \times e^{\frac{i\pi m}{3}} = e^{\frac{i\pi(n+m)}{3}}$

So $f(e^{i\pi n/3} \times e^{i\pi m/3}) = n+m$. So f is a homomorphism. Further, f is clearly a bijection. So f is an isomorphism and C_6 and $\mathbb{Z}_6$ are isomorphic.

No subgroup of $\mathbb{C}$ is isomorphic to S_3. Both addition and multi lication of complex numbers is commutative, but S_3 is non-abelian, for example

$$(1\ 2)(2\ 3) = (1\ 2\ 3)$$
$$(2\ 3)(1\ 2) = (1\ 3\ 2).$$

STEP III 1994 Q8

$$\det\left[\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}\begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix}\right]$$

$$= \det\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}\det\begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix}$$

$$= \left(|z_1|^2 + |z_2|^2\right)\left(|z_3|^2 + |z_4|^2\right) \qquad (*)$$

Set $z_1 = a+bi$, $z_2 = c+di$, $z_3 = p+qi$, $z_4 = r+si$, then

$$(*) = (a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2)$$

Further, $\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}\begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix} = \begin{pmatrix} z_1z_3 - z_2z_4^* & z_1z_4 + z_2z_3^* \\ -z_1^*z_4^* - z_2^*z_3 & z_1^*z_3^* + z_2^*z_4 \end{pmatrix}$

$$= \begin{pmatrix} w_1 & w_2 \\ -w_2^* & w_1^* \end{pmatrix}, \quad \text{where } \begin{matrix} w_1 = z_1z_3 - z_2z_4^* \\ w_2 = z_1z_4 + z_2z_3^* \end{matrix}$$

and $\det\begin{pmatrix} w_1 & w_2 \\ -w_2^* & w_1^* \end{pmatrix} = |w_1|^2 + |w_2|^2$

$$\begin{aligned} w_1 &= (a+bi)(p+qi) - (c+di)(r-si) \\ &= ap + aqi + bpi - bq - cr + csi - dri - ds \\ &= (ap - ds - bq - cr) + i(aq + bp + cs - dr) \end{aligned}$$

$$\begin{aligned} w_2 &= (a+bi)(r+si) + (c+di)(p-qi) \\ &= ar + asi + bri - bs + cp - cqi + dpi + dq \\ &= (ar + cp + dq - bs) + i(as + br + dp - cq) \end{aligned}$$

So, $(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2) =$

$$(ap-ds-bq-cr)^2 + (aq+bp+cs-dr)^2 + (ar+cp+dq-bs)^2 + (as+br+dp-cq)^2$$

STEP III 1994 Q9

Conservation of energy:

$$E = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 + mgy = \text{constant}$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 + 2gy = \text{constant.}$$

And so $\left(\frac{ds}{dt}\right)^2 + \frac{gs^2}{2k} = 0$

Differentiating w.r.t. t,

$$2\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right) + \frac{gs}{k}\left(\frac{ds}{dt}\right) = 0$$

$$\Rightarrow \frac{d^2s}{dt^2} = -\frac{g}{2k}s$$

This is SHM, and initial velocity is 0, and $s(0) = 2\sqrt{kh}$, so

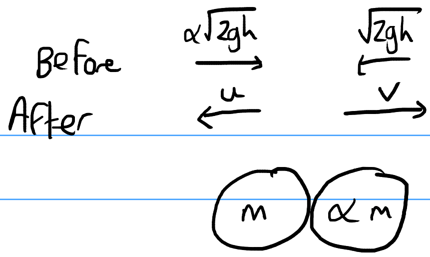
$$s(t) = 2\sqrt{kh}\cos\left(\sqrt{\frac{g}{2k}}\,t\right)$$

The particle reaches V when $\sqrt{\frac{g}{2k}}\,t = \frac{\pi}{2}$

$$\Rightarrow t = \pi\sqrt{\frac{k}{2g}}, \text{ as required.}$$

Note $\dot{s}(t) = \sqrt{2gh}\left(\sqrt{\frac{g}{2k}}\,t\right)$, so the speed of a particle at V, if released at rest from height h, is $\sqrt{2gh}$.

So, the particle with mass m has speed $\alpha\sqrt{2gh}$, and the other particle has speed $\sqrt{2gh}$ when they collide (they collide at V because the time to reach V is independent of h).



Conservation of momentum tells us

$$\alpha m\sqrt{2gh} - \alpha m\sqrt{2gh} = -mu + \alpha m v$$
$$\Rightarrow u = \alpha v$$

Law of restitution tells us

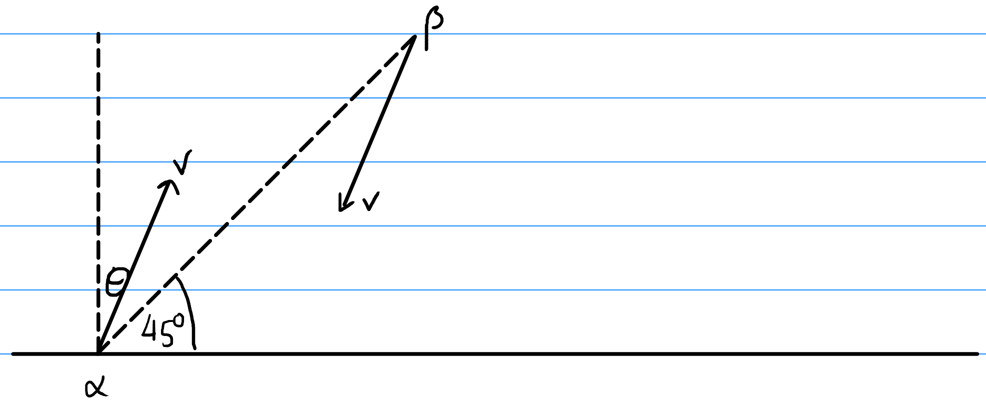
$$u + v = \alpha(1+\alpha)\sqrt{2gh}$$

Substituting, $(1+\alpha)v = \alpha(1+\alpha)\sqrt{2gh}$

$$\Rightarrow v = \alpha\sqrt{2gh}, \quad u = \alpha^2\sqrt{2gh}$$

These are the velocities they had prior to the collision, reduced by a factor of α. Note that speed at collision is proportional to $\sqrt{h}$, so the height the particles rebound to is reduced by a factor of α^2. So we see repeated collisions at V, with the height the particles rebound to decreased by a factor α^2 each time.

STEP III 1994 Q10



The plane's velocity is $\begin{pmatrix} w+v\sin\theta \\ v\cos\theta \end{pmatrix}$. This must be parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so

$$w + v\sin\theta = v\cos\theta$$
$$\Rightarrow \cos\theta - \sin\theta = \frac{w}{v}$$

For the return journey the initial velocity is $\begin{pmatrix} w-v\sin\theta \\ -v\cos\theta \end{pmatrix}$. Reaching the coast takes t hours (as the $\underline{j}$ component is just the negative of the previous).

When we reach the south coast, our horizontal displacement from β is $t(w - v\sin\theta)$. Note the total vertical (and horizontal) distance from α to β is $tv\cos\theta$, and so the distance we are now from α is $tv\cos\theta + t(w - v\sin\theta)$

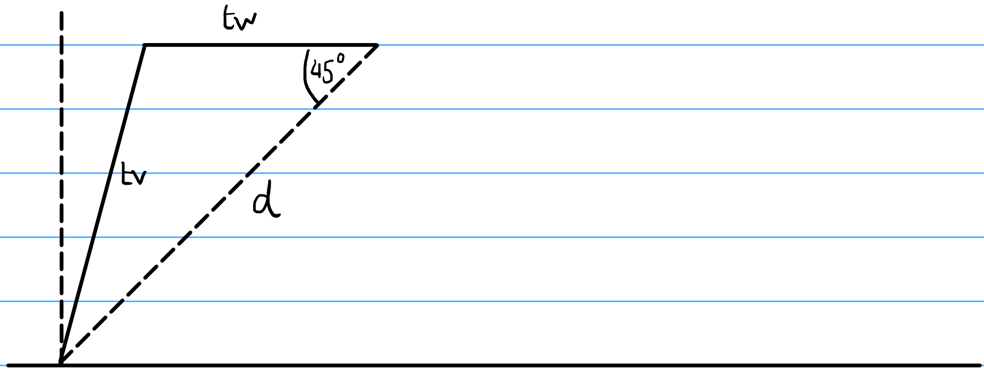
$$= tv(\cos\theta - \sin\theta) + tw$$
$$= tv\left(\frac{w}{v}\right) + tw$$
$$= 2tw.$$

Once we turn west, our speed is $v - w$, so the time taken is $\frac{2tw}{v-w}$. So the total time taken is

$$\frac{2tw}{v-w} + t$$

$$= t\left(\frac{2w + v - w}{v - w}\right)$$

$$= \left(\frac{v+w}{v-w}\right)t, \text{ as required.}$$



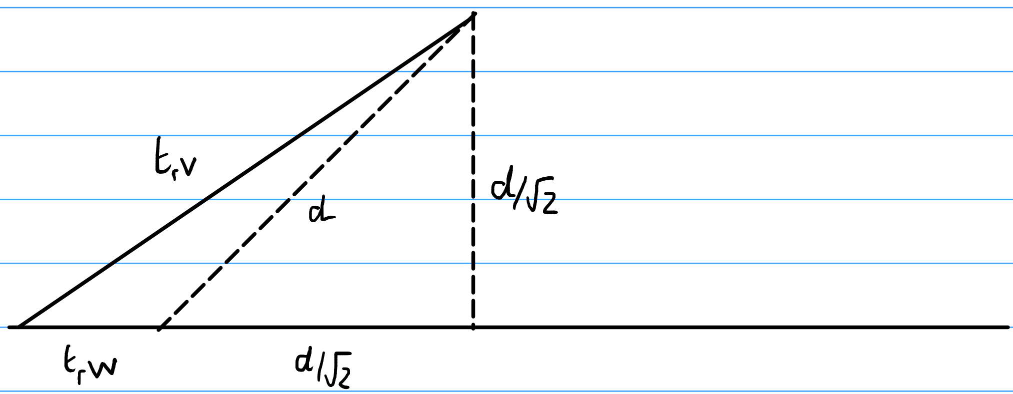
Using the cosine rule,

$$(tv)^2 = (tw)^2 + d^2 - 2tdw \cdot \cos(45^\circ)$$

$$\Rightarrow t(v^2 - w^2) + \sqrt{2}wdt - d^2 = 0$$

$$\Rightarrow t = \frac{-\sqrt{2}wd + \sqrt{2w^2d^2 + 4(v^2-w^2)d^2}}{2(v^2-w^2)}$$

$$= \frac{-\sqrt{2}wd + d\sqrt{4v^2 - 2w^2}}{2(v^2-w^2)}$$



Letting t_r be the time to return to d, we obtain

$$(t_r v)^2 = (t_r w + d/\sqrt{2})^2 + (d/\sqrt{2})^2$$

$$\Rightarrow t_r^2 v^2 = t_r^2 w^2 + 2dwt_r + d^2/2 + d^2/2$$

$$\Rightarrow t_r^2(v^2-w^2) - \sqrt{2}dwt_r - d^2 = 0$$

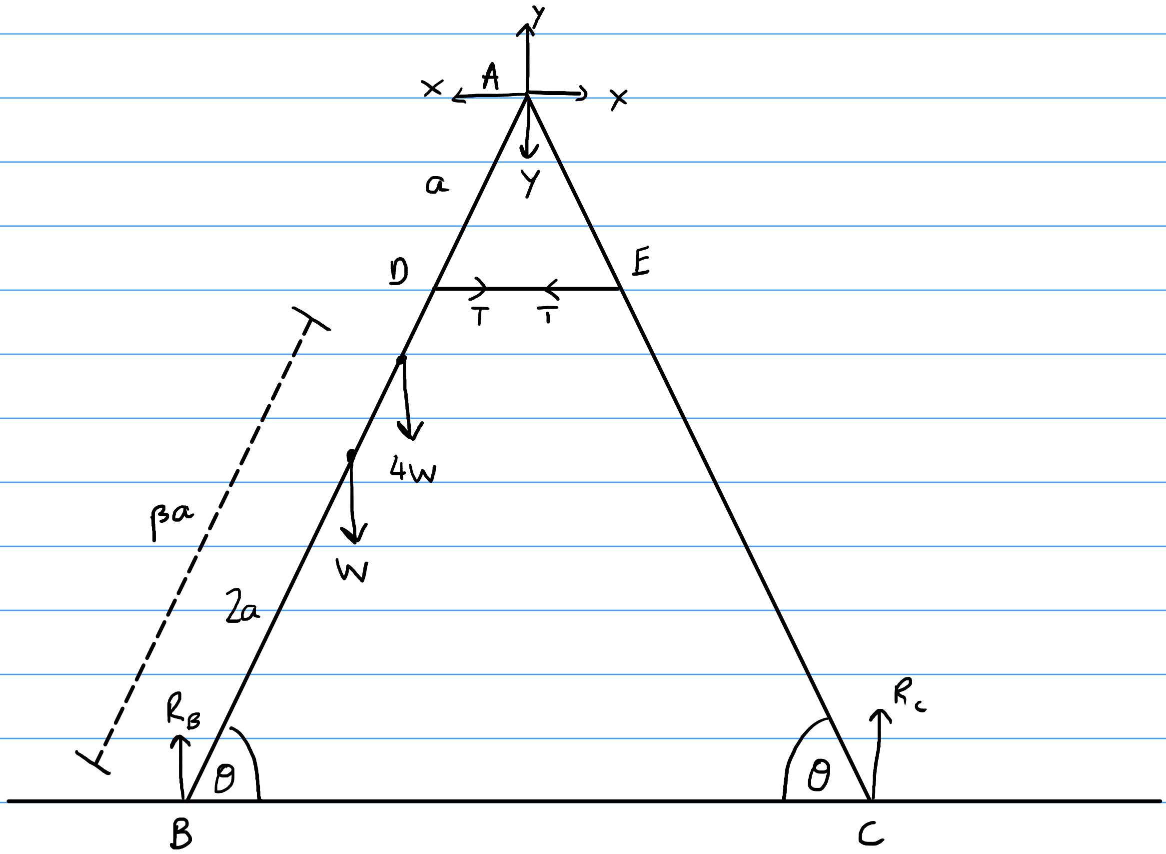
$$\Rightarrow t_r = \frac{\sqrt{2}dw + \sqrt{2w^2d^2 + 4d^2(v^2-w^2)}}{2(v^2-w^2)}$$

$$= \frac{\sqrt{2}dw + d\sqrt{4v^2-2w^2}}{2(v^2-w^2)}$$

$$= t + 2 \cdot \frac{\sqrt{2}dw}{2(v^2-w^2)}$$

$$= t + \frac{\sqrt{2}wd}{v^2-w^2}$$, as required.

STEP III 1994 Q11



Resolving vertically for the whole system, $R_B + R_C = 5W$

Taking moments about B for the whole system,

$$4W\beta\cancel{a}\cancel{\cos\theta} + 2W\cancel{a}\cancel{\cos\theta} = 8\cancel{a}R_C\cancel{\cos\theta}$$

$$\Rightarrow R_C = \frac{2\beta+1}{4}W$$

$$\Rightarrow R_B = 5W - \frac{2\beta+1}{4}W$$
$$= \frac{19-2\beta}{4}W$$

Now taking moments about A for AB,

$$T\cancel{a}\sin\theta + 2\cancel{a}W\cos\theta + 4W(4-\beta)\cancel{a}\cos\theta = 4\cancel{a}R_B\cos\theta$$

$\Rightarrow T\sin\theta = (4R_B - 18W + 4\beta W)\cos\theta$

$= (19W - 2\beta W - 18W + 4\beta W)\cos\theta$

$= (W + 2\beta W)\cos\theta$

$= (1 + 2\beta)W\cos\theta$

But also $T = \frac{\lambda x}{l} = \frac{W(2a\cos\theta - a/4)}{a/4}$

$= (8\cos\theta - 1)W.$

So, $(8\cos\theta - 1)W\sin\theta = (1 + 2\beta)W\cos\theta$

$\Rightarrow \beta\cos\theta = \frac{1}{2}(8\cos\theta\sin\theta - \sin\theta - 1)$

$\Rightarrow \beta = \frac{1}{2}(8\sin\theta - \tan\theta - 1)$

Differentiating w.r.t θ,

$4\cos\theta - \frac{1}{2}\sec^2\theta = 0$

$\Rightarrow \cos^3\theta = 1/8$

$\Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = 60°.$

So $\beta_{max} = \frac{1}{2}(4\sqrt{3} - \sqrt{3} - 1)$

$= \frac{1}{2}(3\sqrt{3} - 1)$

So maximum height $= \beta a \sin 60°$

$= \frac{1}{2}(3\sqrt{3} - 1)a \cdot \frac{\sqrt{3}}{2}$

$= \frac{9 - \sqrt{3}}{4}a.$

STEP III 1994 Q12

When A serves first, we have

$$p_{short} = p_A p_B + p_A q_B p_{short} + q_A p_B p_{short}$$

$$\Rightarrow p_{short} = \frac{p_A p_B}{1 - p_A q_B - q_A p_B}$$

Note that this is symmetric in A and B, so it doesn't matter if B serves first, the result is the same.

$$P(\text{decided in 4 games}) = P(4-0) + P(0-4) + P(3-1) + P(1-3)$$

$$= p_A^2 p_B^2 + q_A^2 q_B^2 + 2p_A^2 p_B q_B + 2p_A q_A p_B^2 + 2q_A^2 q_B p_B + 2q_A p_A q_B^2$$

$$= p_A^2 p_B^2 + q_A^2 q_B^2 + 2(p_A p_B + q_A q_B)(p_A q_B + q_A p_B) \text{, as required.}$$

Now, $p_{ston} = p^4 + 4p^3 q + 6p^2 q^2 p_{short}$

Note $1-2pq$
$= 1-2p(1-p)$
$= p^2 + p^2 - 2p + 1$
$= p^2 + (1-p)^2$
$= p^2 + q^2$
$= 2q^2 - 2q + 1$

So $p_{ston} - p_{short} = p^2\left[p^2 + 4pq + \frac{6p^2q^2 - 1}{1-2pq}\right]$

$$= \frac{p^2}{p^2+q^2}\left[(2q^2-2q+1)(1-q)^2 + 4(1-q)q(2q^2-2q+1) + 6(1-q)^2q^2 - 1\right]$$

$$= \frac{p^2}{p^2+q^2}\left[(2q^2-2q+1)(1-2q+q^2) + (4q-q^2)(2q^2-2q+1) + 6(1-2q+q^2)q^2 - 1\right]$$

$$= \frac{p^2}{p^2+q^2}\left[q^4(2-8+6) + q^3(-2-4+8+8-12) + q^2(1+2+4-4-8+6) + (1-1)\right]$$

$$= \frac{p^2}{p^2+q^2} \cdot (-2q^3 + q^2)$$

$$= \frac{p^2q^2}{p^2+q^2}(1-2q) = \frac{p^2q^2}{p^2+q^2}(p-q) \text{, as required.}$$

STEP III 1994 Q13

Note $P(T \le t) = \int_0^t \lambda e^{-\lambda u}\,du$
$= \left[-e^{-\lambda u}\right]_0^t$
$= 1-e^{-\lambda t}$

So $P(T \le t_0 + t \mid T \geqslant t_0)$

$$= \frac{P(t_0 \le T \le t+t_0)}{P(T \geqslant t_0)} = \frac{1-e^{-\lambda(t+t_0)} - (1-e^{-\lambda t_0})}{e^{-\lambda t_0}}$$

$$= \frac{e^{-\lambda t_0}(1-e^{-\lambda t})}{e^{-\lambda t_0}}$$

$$= 1-e^{-\lambda t}$$
$$= P(T \le t).$$

So, no matter how long the rope has been in use without failure, the time to the failure from the current time is still exponential with parameter λ. This is the memoryless property of the exponential distribution.

$P(\text{both ropes fail by time } t) = P(T_1 \le t \cap T_2 \le t)$
$= P(T_1 \le t)P(T_2 \le t)$ by independence
$= (1-e^{-\lambda t})^2$

So the density function is
$$\frac{d}{dt}(1-e^{-\lambda t})^2$$
$= 2\lambda e^{-\lambda t}(1-e^{-\lambda t})$, as required.

$P(\text{both ropes fail during } n^{th} \text{ performance}) = P(\text{both ropes survive to } (n-1)h \text{ and fail by } nh)$
$= P(\text{first rope survives to } (n-1)h \text{ and fails by } nh)^2$ as they are iid
$= \left[1-e^{-\lambda nh} - (1-e^{-\lambda(n-1)h})\right]^2$
$= (e^{-\lambda(n-1)h} - e^{-\lambda nh})^2$
$= e^{-2n\lambda h}(e^{\lambda h}-1)^2$

So P(both fail on same performance)

$$= \sum_{n=1}^{\infty} e^{-2n\lambda h}(e^{\lambda h}-1)^2$$

$$= (e^{\lambda h}-1)^2 \cdot \frac{e^{-2\lambda h}}{1-e^{-2\lambda h}}$$

$$= \frac{(1-e^{-\lambda h})^2}{(1+e^{-\lambda h})(1-e^{-\lambda h})}$$

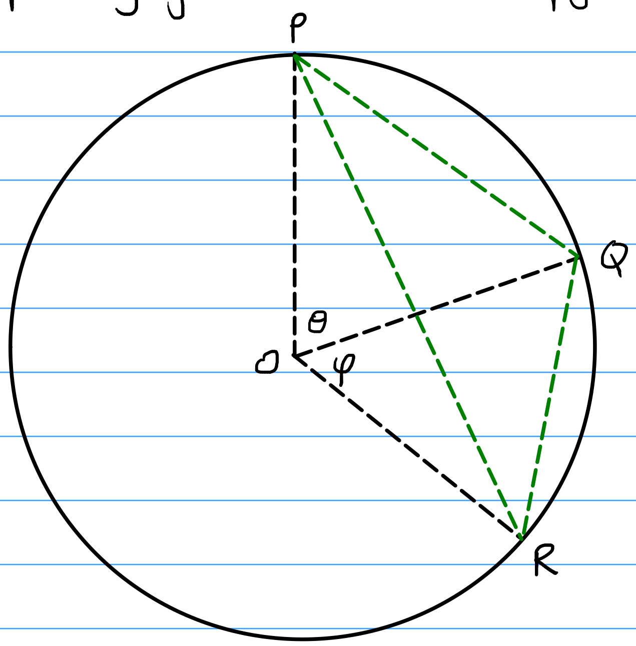
$$= \frac{1-e^{-\lambda h}}{1+e^{-\lambda h}}$$

$$= \frac{e^{\lambda h/2}-e^{-\lambda h/2}}{e^{\lambda h/2}+e^{-\lambda h/2}}$$

$$= \tanh\left(\frac{\lambda h}{2}\right)$$

STEP III 1994 Q14

We begin by considering the case $\frac{1}{2} \le k \le 1$. So the largest angle in the triangle is at least $\pi/2$. For this to happen, all three points must be in the same semicircle (as the angle in a semicircle is $\pi/2$). We will assume Q is the largest angle (which divides the probability by 3), and assume they are in the order PQR in the clockwise direction (which divides the probability by 2). So we need to multiply our final answer by 6.



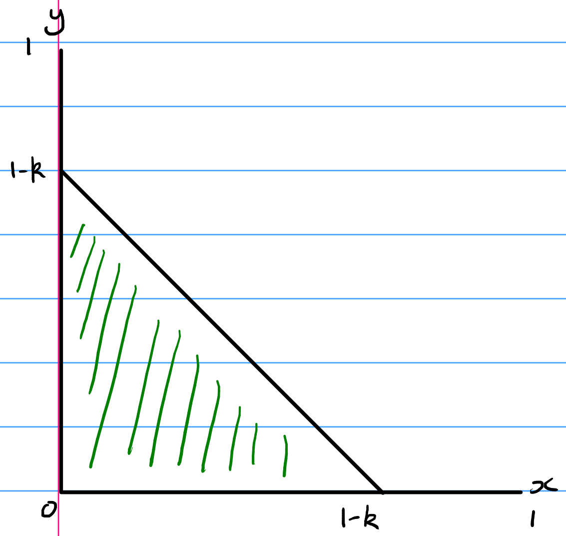
We have $\theta, \varphi \sim U[0, 2\pi]$ indepently.
Further, as OPQ and OQR are isosceles, we have $\angle PQR = \frac{1}{2}(\pi - \theta) + \frac{1}{2}(\pi - \varphi)$
$= \pi - \frac{1}{2}(\theta + \varphi)$.

We want to find $P(\pi - \frac{1}{2}(\theta + \varphi) \geq k\pi)$

Set $x = \frac{\theta}{2\pi}$ and $y = \frac{\varphi}{2\pi}$, so $x, y \sim U[0,1]$ independently.

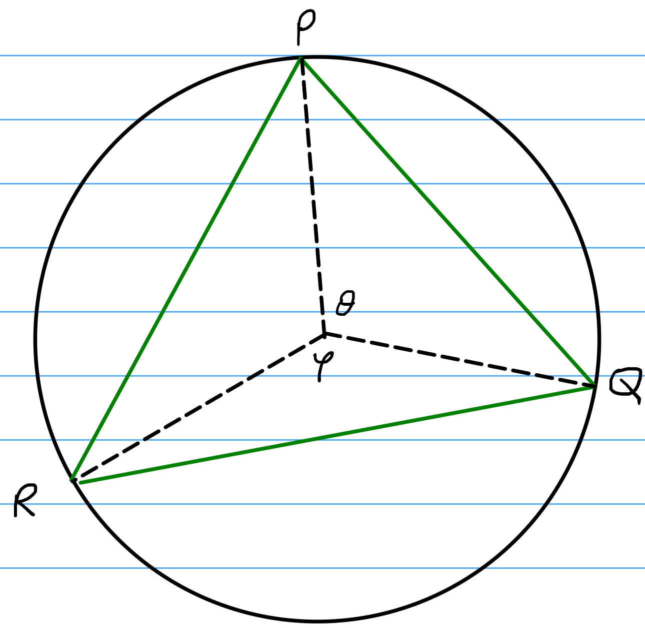
So the probability becomes

$$P(1 - x - y \geq k)$$
$$= P(x + y \leq 1 - k)$$



The probability is equal to the shaded area, which is $\frac{1}{2}(1-k)^2$. Multiplying by 6 as explained above, we obtain $3(1-k)^2$, as required.

Now consider the case $\frac{1}{3} \leq k \leq \frac{1}{2}$. This time, any angle could be the biggest. The only restriction we make is that the points are in the order PQR, so we will need to multiply our answer by 2.



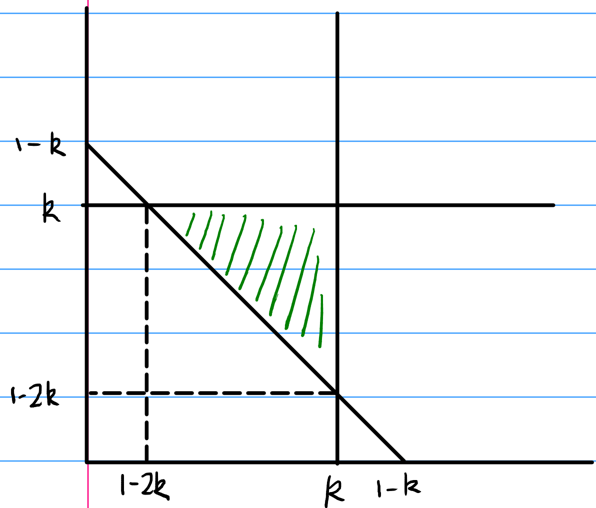
As before, $\theta, \varphi \sim U[0, 2\pi]$ independently.

$\angle PQR = \pi - \frac{1}{2}(\theta + \varphi)$, $\angle QRP = 2\theta$, $\angle RPQ = 2\varphi$.

So $P(\text{all three angles are} \leq k\pi) = P(\pi - \frac{1}{2}(\theta + \varphi) \leq k\pi \cap \frac{\theta}{2} \leq k\pi \cap \frac{\varphi}{2} \leq k\pi)$

As before, set $x = \frac{\theta}{2\pi}$, $y = \frac{\varphi}{2\pi}$

$= P(x + y \geq 1 - k \cap x \leq k \cap y \leq k)$



The probability is the green shaded area, which is $\frac{1}{2} \times \left[k-(1-2k)\right]^2$

$= \frac{1}{2}(3k-1)^2$

As explained above, we must multiply by 2.

So $P(\text{all angles} \leq k\pi) = (3k-1)^2$

so $P(\text{at least one angle} \geq k\pi) = 1-(3k-1)^2$

$\left.\begin{array}{l} \\ \\ \end{array}\right\}$ for $\frac{1}{3} \leq k \leq \frac{1}{2}$