STEP II 1994 Q1

The final digit of $n \times m$ depends only on the final digits of n & m. So, we need to show that for any final digit of n from $\{1, 3, 7, 9\}$, we can choose m so that the product ends in 1.

Final digit of n	m	product	ends in 1?
1	1	1	✓
3	7	21	✓
7	3	21	✓
9	9	81	✓

Now, we want to show that for any n, we can pick m s.t. $n \times m$ ends in a 1, or a 1 followed by 0s.

If n ends in 1, 3, 7, or 9, we are done by the above. Otherwise, n must have at least one factor which is a 2 or 5. So, pick m_1 such that $n \times m_1$ has an equal number of 2s and 5s in its prime factorisation. Suppose $n \times m_1 = 2^k \times 5^k \times A$, where A has no prime factors of 2s or 5s. Then A must end in 1, 3, 7, or 9. Thus $nm_1 = \underbrace{A0000\ldots0}_{k \text{ zeroes}}$. Then we can pick m_2 s.t. $A \times m_2$ ends in 1. So pick $m = m_1 m_2$ and we are done.

Now, n is a k digit number ending in a 1 (or a 1 followed by some 0s).

Choose $M = 9\underbrace{00\ldots0}_{k \text{ zeroes}}8\underbrace{00\ldots0}_{k \text{ zeroes}\ldots}700\ldots \quad 0100\ldots\ldots0$

$Mn = 9 \times A \;\; 8 \times A \;\; 7 \times A \;\; 6 \times A \ldots 1 \times A \; 0 \times A$ (as digits),

and $x \times A$ ends in an x, so Mn contains all the digits from 0 to 9.

STEP II 1994 Q2

$P(x) = (x-a)^m Q(x)$

$P'(x) = m(x-a)^{m-1}Q(x) + (x-a)^m Q'(x)$

$= (x-a)^{m-1}(mQ(x) + (x-a)Q'(x))$

$= (x-a)^{m-1}R(x)$

By induction, $P^{(r)}(x) = (x-a)^{m-r}R(x) \quad (r \leq m-1)$

so $P^{(r)}(a) = (a-a)^{m-r}R(x)$

$= 0 \quad$ For $1 \leq r \leq m-1$

$P_n(x) = \frac{d^n}{dx^n}(x^2-1)^n$. Now $(x^2-1)^n$ is a polynomial of degree x^{2n}, so $P_n(x)$ is a polynomial of degree $(2n-n) = n$.

$\int_{-1}^{1} x^m P_n(x)\,dx \qquad u\ x^m \qquad v'\ \frac{d^n}{dx^n}(x^2-1)^n$

$u'\ mx^{m-1} \qquad v\ \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$

$= \left[x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} - m\int_{-1}^{1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$

The first term is zero, because (x^2-1) is zero for $x=1$ & $x=-1$, and we are only differentiating $(n-1)$ times, so at least one (x^2-1) exists in each term.

We can repeat the integration by parts, and the boundary term stays zero each time. As long as $m \leq n-1$, we end up with

$(-1)^m m! \int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx$

$(-1)^m m! \left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right]_{-1}^{1}$

$= 0$ For the same reason as before.

$$\int_{-1}^{1} x^n P_n(x)\, dx$$

Proceeding via integration by parts, the boundary terms vanish as before, and we are left with

$$(-1)^n n! \int_{-1}^{1} (x^2-1)^n \, dx$$

Use the substitution $x = \sin u$, $dx = \cos u \, du$

$$= (-1)^n n! \int_{-\pi/2}^{\pi/2} (\sin^2 u - 1)^n \cos u \, du$$

$$= (-1)^n n! \int_{-\pi/2}^{\pi/2} (1-\cos^2 u)^n \cos u \, du$$

$$= (-1)^n (-1)^n n! \int_{-\pi/2}^{\pi/2} \cos^{n+1} u \, du$$

$$= 2n! \int_{0}^{\pi/2} \cos^{n+1} u \, du \qquad \text{(as it is an even function)}$$

$$= 2n! \cdot \frac{2^{2n}(n!)^2}{(2n+1)!}$$

$$= \frac{2^{2n+1}(n!)^3}{(2n+1)!}$$

STEP II 1994 Q3

$$f(x-y) = f(x)f(y) - f(a-x)f(a+y) \qquad \forall x, y$$

$f(0)=1$

$x=0, y=0$

$f(0-0) = f(0)f(0) - f(a)f(a)$

$1 = 1 - f(a)^2$

$\Rightarrow f(a)^2 = 0$

i) $x=0, y=t$

$f(0-t) = f(0)f(t) - \underbrace{f(a)}_{=0}f(a+t)$

$f(-t) = f(t)$

ii) $x=0, y=a$

$f(a-a) = f(a)f(a) - f(0)f(2a)$

$1 = 0 - f(2a)$

$f(2a) = -1$

iii) $x=2a, y=t$

$=0$ as $f(-a) = f(a) = 0$

$f(2a-t) = f(2a)f(t) - f(-a)f(a+t)$

$f(2a-t) = -1 \times f(t) - 0$

$f(2a-t) = -f(t)$

iv) $x=2a, y=-2a-t$

$f(2a-(-2a-t)) = f(2a)f(-2a-t) - f(-a)f(-a+t)$

$f(4a+t) = -f(2a+t) + 0$

Now $f(2a+t) = f(2a-(-t)) = f(2a)f(-t) - f(-a)f(a-t)$

$= -f(t)$

So $f(4a+t) = -(-f(t))$
$= f(t)$

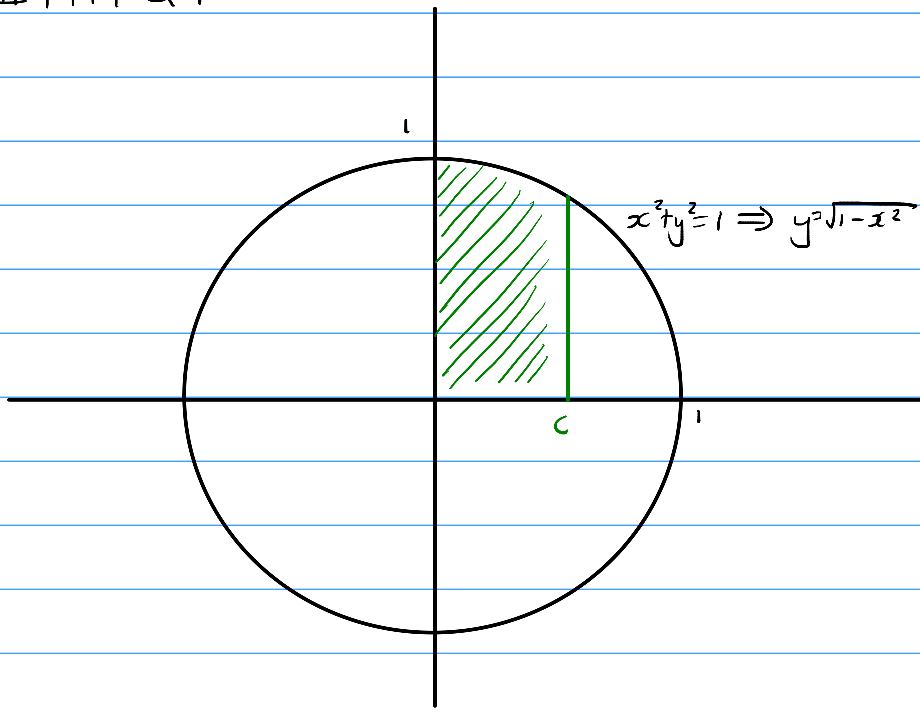
So, this function is periodic, even, and $f(\pi/2) = 0$. Try $f(x) = \cos x$.

$$\begin{aligned}
\cos(x-y) &= \cos x \cos y + \sin x \sin y \\
&= \cos x \cos y + \cos(\pi/2 - x)\cos(\pi/2 - y) \\
&= \cos x \cos y - \cos(\pi/2 - x)\cos(\pi/2 + y) \\
&= f(x)f(y) - f(\pi/2 - x)f(\pi/2 + y), \text{ as required.}
\end{aligned}$$

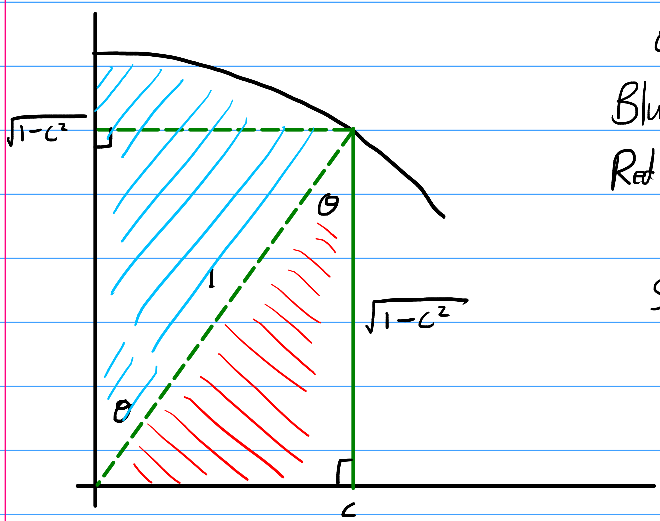
Now choose $a = -2$. We want to stretch $\cos x$ so the root at $-\pi/2$ is now at -2, this is a stretch by a factor of $4/\pi$, so $f(x) = \cos(\frac{\pi}{4}x)$.

$$\begin{aligned}
\cos(\tfrac{\pi}{4}(x-y)) &= \cos\tfrac{\pi}{4}x \cos\tfrac{\pi}{4}y + \sin\tfrac{\pi}{4}x \sin\tfrac{\pi}{4}y \\
&= f(x)f(y) + \sin(\tfrac{\pi}{2} - \tfrac{\pi}{4}x)\sin(\tfrac{\pi}{2} - \tfrac{\pi}{4}y) \\
&= f(x)f(y) + \cos(\tfrac{\pi}{2} - \tfrac{\pi}{4}x)\cos(\pi/2 - \pi/4\, y) \\
&= f(x)f(y) - \cos(\tfrac{\pi}{2} - \tfrac{\pi}{4}x)\cos(\tfrac{\pi}{2} + \tfrac{\pi}{4}y) \\
&= f(x)f(y) - f(a - x)f(a + y), \text{ as required.}
\end{aligned}$$

STEP II 1994 Q4



$\int_0^c (1-x^2)^{1/2}\,dx$ is the green shaded area.



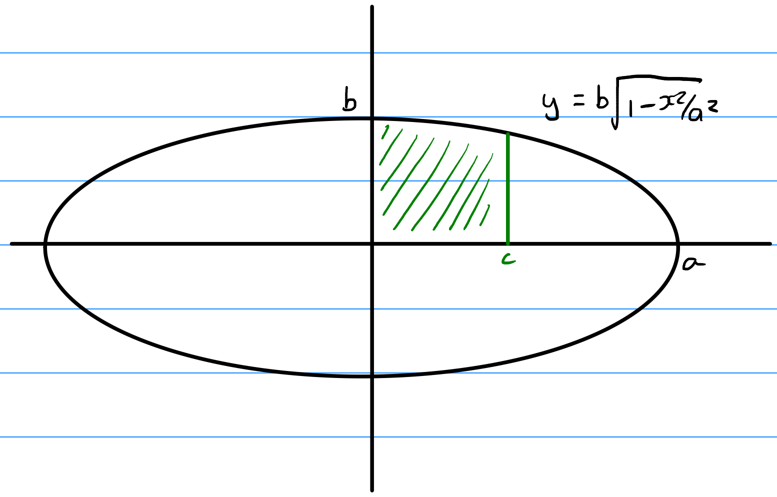
$\theta = \sin^{-1}(c)$

Blue area is $\frac{1}{2}r^2\theta = \frac{1}{2}\times 1^2 \times \sin^{-1}(c)$

Red area is $\frac{1}{2}ab\sin C = \frac{1}{2}\times 1 \times \sqrt{1-c^2} \times \sin\theta$

$= \frac{1}{2}\sqrt{1-c^2}\cdot c$

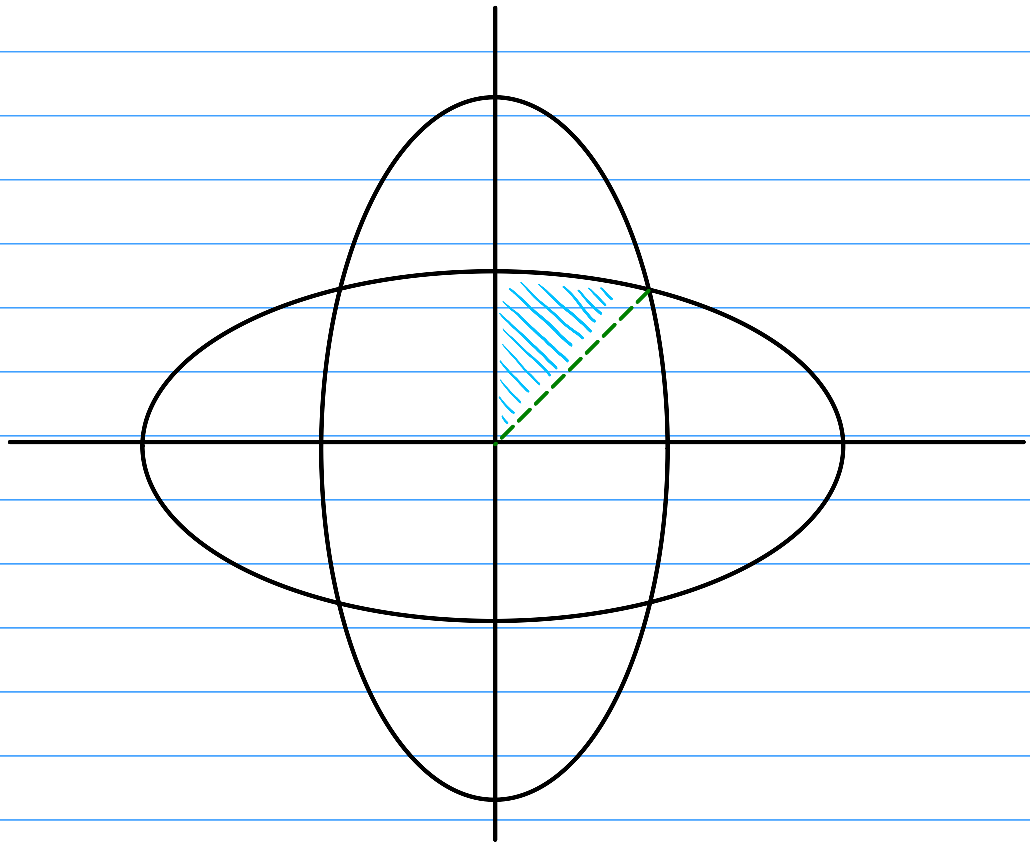
So total area is $\frac{1}{2}\left(c\sqrt{1-c^2}+\sin^{-1}(c)\right)$, as required.



$$\text{Area} = \int_0^c b\sqrt{1-x^2/a^2}\,dx \qquad u = x/a, \quad dx = a\,du$$

$$= ab\int_0^{c/a} \sqrt{1-u^2}\,du$$

$$= \frac{ab}{2}\left(\frac{c}{a}(1-c^2/a^2)^{1/2} + \sin^{-1}\frac{c}{a}\right)$$, using the previous result.



By symmetry, the area we want to find is 8 times the blue area.

To find the point of intersection:

$$b^2(1-x^2/a^2) = a^2(1-x^2/b^2)$$

$$b^2-a^2 = x^2\left(\frac{b^2}{a^2}-\frac{a^2}{b^2}\right)$$

$$b^2-a^2 = x^2\left(\frac{b^4-a^4}{a^2b^2}\right)$$

$$\Rightarrow x^2 = \frac{a^2b^2(b^2-a^2)}{(b^2-a^2)(b^2+a^2)}$$

$$= \frac{a^2b^2}{b^2+a^2}$$

So the blue shaded area is the 'sin^{-1}' part of the previous answer (this was the bit from the sector of the circle), with $c=\frac{ab}{\sqrt{a^2+b^2}}$

So, the blue shaded area is $\frac{ab}{2}\sin^{-1}\left(\frac{ab}{\sqrt{a^2+b^2}} \div a\right)$

$$= \frac{ab}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

Multiplying by 8, we obtain $4ab\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)$, as required.

STEP II 1994 Q5

i) $(x-1)^4+(x+1)^4=c$

$x^4-\cancel{4}x^3+6x^2-\cancel{4x}+1+x^4+\cancel{4}x^3+6x^2+\cancel{4x}+1=c$

$\Rightarrow 2x^4+12x^2+(2-c)=0$

$\Rightarrow x^2=\dfrac{-12\pm\sqrt{144-8(2-c)}}{2}$

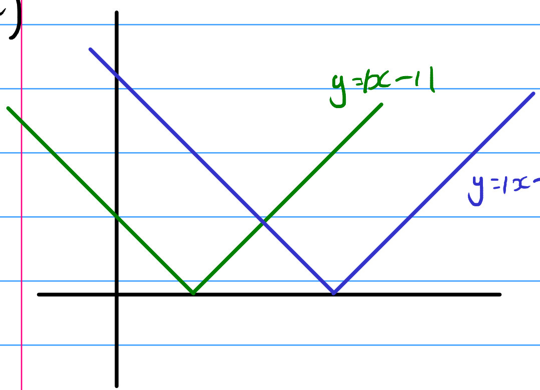
The negative solution gives no real solutions for x. For the positive solution, there are no solutions if $8(2-c)>0$, one solution if $8(2-c)=0$, two solutions if $8(2-c)<0$

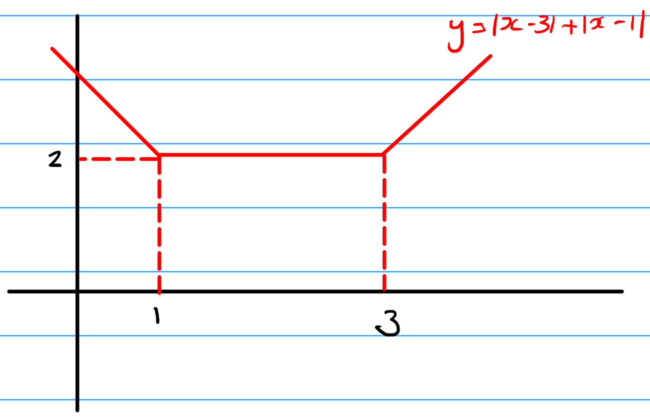
$\Leftrightarrow$ two real roots if $c>2$, one root if $c=2$, no real roots if $c<2$.

ii) $(x-3)^4+(x-1)^4=c$

Set $u=x-2$ so $(u-1)^4+(u+1)^4=c$, so we have the same number of solutions depending on the value of c as before.

iii)





So there are 2 solutions if $c>2$, infinite solutions if $c=2$, and 0 solutions if $c<2$.

iv) $(x-3)^3+(x-1)^3=c$

Set $u=x-2$. Then

$(u-1)^3+(u+1)^3=c$

$\Rightarrow u^3-\cancel{3u^2}+3u\cancel{-1}+u^3+\cancel{3u^2}+3u\cancel{+1}=c$

$\Rightarrow u^3+3u-c=0$

Set $y=u^3+3u-c$, then $\frac{dy}{du}=3u^2+3$

$=3(u^2+1)>0\ \forall u.$

So this cubic has no turning points, and so crosses the x-axis exactly once.

So the equation has exactly one solution for u and hence for x for all values of c.

STEP II 1994 Q6

Proceed via induction.

Base case

We want to show $\frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} - \cot\theta$

Using $\tan\theta/2 = t$, $RHS = \frac{1}{2t} - \frac{(1-t^2)}{2t}$

$$= \frac{t^2}{2t}$$

$$= \frac{1}{2}t, \text{ as required.}$$

Assumption

Assume true for $n=k$.

Induction

For $n=k+1$, $LHS = \underbrace{\frac{1}{2}\tan\frac{\theta}{2} + \dots + \frac{1}{2^k}\tan\frac{\theta}{2^k}}_{=f(k)} + \frac{1}{2^{k+1}}\tan\frac{\theta}{2^{k+1}}$

$$= \frac{1}{2^k}\cot\frac{\theta}{2^k} - \cot\theta + \frac{1}{2^{k+1}}\tan\frac{\theta}{2^{k+1}}$$

Set $t = \tan\frac{\theta}{2^{k+1}}$, then $= \frac{1}{2^k}\cdot\frac{1-t^2}{2t} - \cot\theta + \frac{1}{2^{k+1}}t$

$$= \frac{1}{2^{k+1}}\left(\frac{1-t^2}{t} + t\right) - \cot\theta$$

$$= \frac{1}{2^{k+1}}\cdot\frac{1}{t} - \cot\theta$$

$$= \frac{1}{2^{k+1}}\cot\frac{\theta}{2^{k+1}} - \cot\theta, \text{ as required.}$$

True for $n=1$ and if true for $n=k$, then true for $n=k+1$, so true $\forall k \in \mathbb{N}$.

$$\sum_{r=1}^{\infty}\frac{1}{2^r}\tan\frac{\theta}{2^r} = \lim_{n\to\infty}\frac{1}{2^n}\cot\frac{\theta}{2^n} - \cot\theta$$

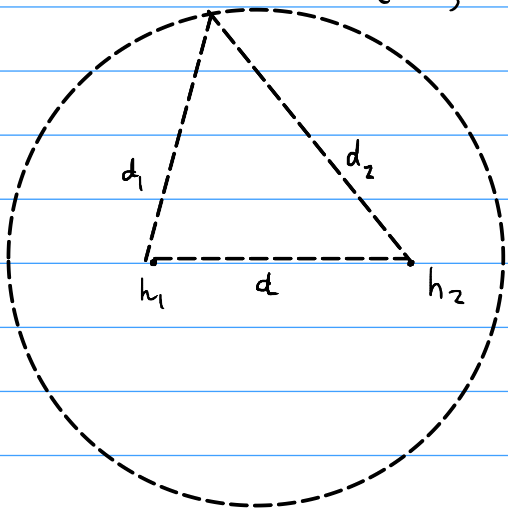
$$= \lim_{n\to\infty}\frac{1}{2^n}\cdot\frac{2^n}{\theta} - \cot\theta \quad \text{(Small angle approximation)}$$

$$= \theta - \cot\theta, \text{ as required.}$$

STEP II 1994 Q7

$$x^2+y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0$$
$$\Rightarrow \left(x+g/a\right)^2+\left(y+f/a\right)^2=\frac{g^2+f^2-ac}{a^2}$$

is a circle with centre $\left(-g/a, -f/a\right)$ and radius $\frac{\sqrt{g^2+f^2-ac}}{a}$.



Keeping the angles of elevation the same is equivalent to keeping $\frac{h_1}{d_1}=\frac{h_2}{d_2}$. $(*)$

Suppose the flagpole with height h_1 is at $(0,0)$, and the flagpole with height h_2 is at $(0,d)$. Then $(*)$ becomes

$$\frac{h_1}{\sqrt{x^2+y^2}}=\frac{h_2}{\sqrt{(x-d)^2+y^2}}$$

So $h_1^2\left((x-d)^2+y^2\right)=h_2^2\left(x^2+y^2\right)$
$$\Rightarrow (h_1^2-h_2^2)x^2+(h_1^2-h_2^2)y^2-2h_1^2dx+h_1^2d^2=0$$
Assume wlog $h_1>h_2$, then we can use the first result to say this is a circle with centre $\left(\frac{2h_1^2d}{h_1^2-h_2^2}, 0\right)$.

If the two flagpoles have the same height, then the soldier marches along the perpendicular bisector to the line segment joining the two flagpoles.

Now we consider three flagpoles, with heights h_i. Suppose each has coordinates (x_i, y_i).

So, by the same logic as before,
$$\frac{h_i^2}{(x-x_i)^2+(y-y_i)^2}=\frac{h_j^2}{(x-x_j)^2+(y-y_j)^2}$$

$$\Rightarrow h_i^2(x-x_j)^2 + h_i^2(y-y_j)^2 = h_j^2(x-x_i)^2 + h_j^2(y-y_i)^2$$

so $$(h_i^2-h_j^2)x^2 + (h_i^2-h_j^2)y^2 + 2x(h_j^2x_i - h_i^2x_j) + 2y(h_j^2y_i - h_j^2y_j)$$
$$+ h_i^2x_j^2 + h_i^2y_j^2 - h_j^2x_i^2 - h_j^2y_i^2 = 0$$

Which is a circle with centre

$$\left(\frac{h_i^2x_j - h_j^2x_i}{h_i^2-h_j^2}, \frac{h_i^2y_i - h_j^2y_i}{h_i^2-h_j^2}\right) = (x_{ij}, y_{ij})$$

So $h_3^2(h_1^2-h_2^2)x_{12} + h_1^2(h_2^2-h_3^2)x_{23} + h_2^2(h_3^2-h_1^2)x_{31}$
$= h_3^2(h_1^2x_2 - h_2^2x_1) + h_1^2(h_2^2x_3 - h_3^2x_2) + h_2^2(h_3^2x_1 - h_1^2x_2)$
$= 0$, as required.

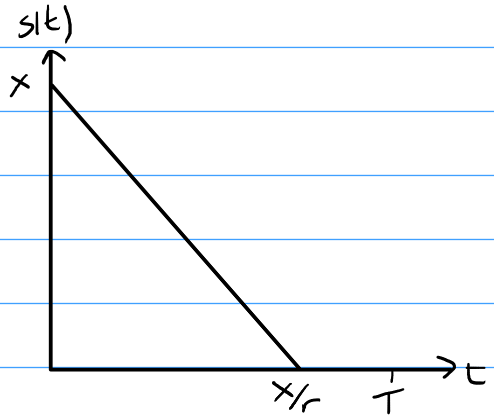
Exactly the same argument follows for the y_i.

Now, $$x_{12} = \frac{h_1^2(h_2^2-h_3^2)x_{23} + h_2^2(h_3^2-h_1^2)x_{31}}{h_3^2(h_2^2-h_1^2)}$$ with similar for y_{12}.

Also, $h_1^2(h_2^2-h_3^2) + h_2^2(h_3^2-h_1^2) = h_3^2(h_2^2-h_1^2)$, so (x_{12}, y_{12}) is a weighted average of (x_{23}, y_{23}) and (x_{31}, y_{31}), so all three points must lie on a straight line.

STEP II 1994 Q8

The average input of items per hour is X/T, and the units sold per hour is r. So, to prevent the stock growing without bound we must have $r \geq X/T$ or $X \leq rT$.



In T hours, costs are

- Purchase costs, at $a+qX$
- Storage costs, at $\frac{1}{2}\cdot X/r \cdot X \cdot b = \frac{1}{2}\frac{X^2b}{r}$
- Running costs, at cT

The sales are $(p+q)X$

So profit in a T hour period is

$$(p+q)X - (a+qX) - \frac{1}{2}\frac{X^2b}{r} - cT$$

$$= pX - a - \frac{1}{2}\frac{X^2b}{r} - cT$$

Now consider this as a quadratic in X. Completing the square,

$$\text{profit} = \frac{-b}{2r}\left(X - \frac{pr}{b}\right)^2 + \frac{b}{2r}\left(\frac{pr}{b}\right)^2 - (a+cT)$$

$$= -\frac{b}{2r}\left(X - \frac{pr}{b}\right) + \frac{p^2r}{2b} - a - cT$$

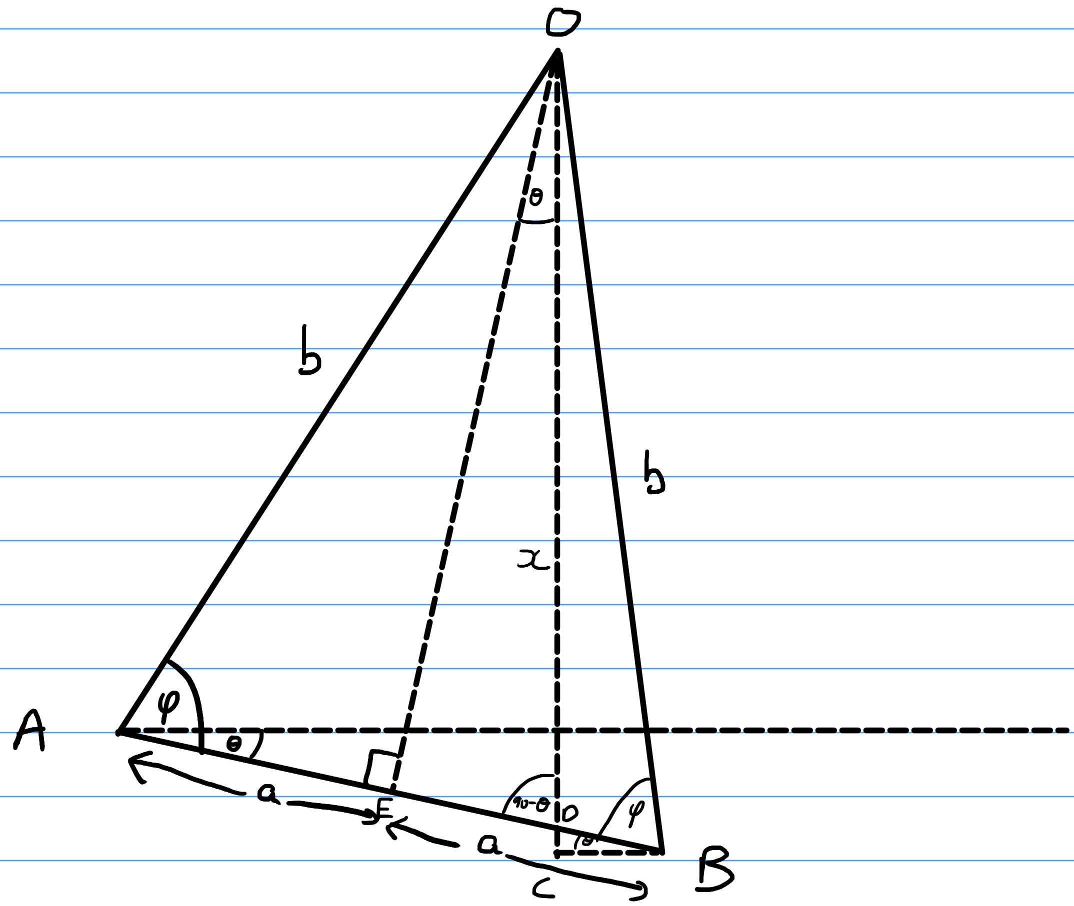
This is maximised by taking $X = \frac{pr}{b}$, so then

$$\text{profit} = \frac{p^2 r}{2b} - (a + cT)$$

To make this positive, we need $\frac{p^2 r}{2b} > a + cT$

$$\Rightarrow p^2 > 2\frac{(a+cT)b}{r}$$, as required.

STEP II 1994 Q9



Consider length x. Using triangle OBC, $x = b\sin(\varphi+\theta)$.
But by splitting line OC at D, we can write $x = OD + DC$.

Now $OE = b\sin\varphi$ (using triangle OAE), so $OD = \frac{b\sin\varphi}{\cos\theta}$ (using triangle OED).

Further, the centre of mass of AB must be at D (to make the moments balance), so we have $AD = \frac{4}{3}a$ and $DB = \frac{2}{3}a$. Thus $DC = \frac{2}{3}a\sin\theta$ (using triangle DBC).

So, $b\sin(\varphi+\theta) = \frac{b\sin\varphi}{\cos\theta} + \frac{2}{3}a\sin\theta$.

So $b\sin\varphi\cos\theta + b\cos\varphi\sin\theta = \frac{b\sin\varphi}{\cos\theta} + \frac{2}{3}a\sin\theta$. $(*)$

But considering triangle OAE (and noting $OE = \sqrt{b^2-a^2}$), we have
$\sin\varphi = \frac{\sqrt{b^2-a^2}}{b}$ and $\cos\varphi = a/b$

So (*) becomes

$$\sqrt{b^2-a^2}\cos\theta + a\sin\theta = \frac{\sqrt{b^2-a^2}}{\cos\theta} + \frac{2}{3}a\sin\theta$$

$$\Rightarrow \frac{1}{3}a\cos\theta = \sqrt{b^2-a^2}\left(\frac{1}{\cos\theta} - \cos\theta\right)$$

$$= \sqrt{b^2-a^2}\left(\frac{1-\cos^2\theta}{\cos\theta}\right)$$

$$= \sqrt{b^2-a^2}\ \frac{\sin^2\theta}{\cos\theta}$$

$$\Rightarrow \frac{1}{3}a = \sqrt{b^2-a^2}\tan\theta$$

$$\Rightarrow \tan\theta = \frac{a}{3\sqrt{b^2-a^2}}$$, as required.

Now consider moments about B.

$$\cancel{2a}mg\cos\theta = \cancel{2a}T\sin\alpha$$

$$\Rightarrow mg\cos\theta = \frac{T\sqrt{b^2-a^2}}{b} \qquad (\dagger)$$

From previously, $\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{a}{3\sqrt{b^2-a^2}}$

$$\Rightarrow \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{a^2}{9(b^2-a^2)}$$

$$\Rightarrow 1-\cos^2\theta = \frac{a^2\cos^2\theta}{9(b^2-a^2)}$$

$$\Rightarrow 1 = \cos^2\theta\left(\frac{a^2}{9(b^2-a^2)} + 1\right)$$

$$= \cos^2\theta\left(\frac{a^2+9b^2-9a^2}{9(b^2-a^2)}\right)$$

$$= \cos^2\theta\left(\frac{9b^2-8a^2}{9(b^2-a^2)}\right)$$

$$\Rightarrow \cos\theta = \frac{3\sqrt{b^2-a^2}}{\sqrt{9b^2-8a^2}}$$

Substituting this into (†),

$$mg \cdot \frac{3\sqrt{b^2-a^2}}{\sqrt{9b^2-8a^2}} = \frac{T\sqrt{b^2-a^2}}{b}$$

$$\Rightarrow T = \frac{3bmg}{\sqrt{9b^2-8a^2}}.$$

STEP II 1994 Q10

$T = \frac{\lambda}{l}x$, where x is the extension.

So, using N2L on the trailer,

$T = -m\ddot{x}$, so $\frac{\lambda}{l}x = -m\ddot{x}$.

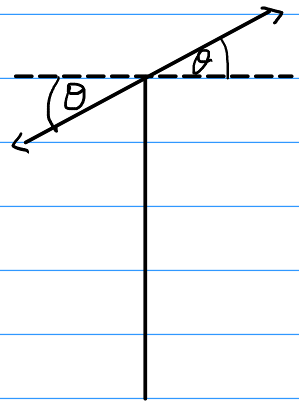
Thus the trailer initially moves with SHM with $\omega = \sqrt{\frac{\lambda}{ml}}$

So, the time taken for the trailer to return to its initial position relative to the truck is $\frac{1}{2}\cdot\frac{2\pi}{\omega} = \pi\sqrt{\frac{lm}{\lambda}}$.

At this point, the rope becomes slack, so the trailer moves with constant speed. Now, the initial speed of the trailer relative to the truck was $-u$, so now its speed is u (as it has undergone SHM). So the time taken to catch up is l/u.

So the total time taken is $\pi\left(\frac{lm}{\lambda}\right)^{1/2} + \frac{l}{u}$, as required.

STEP II 1994 Q11



Considering first the initial flight,

s	?
u	v
v	0
a	-g
t	x

$\Rightarrow v^2 - 2gs = 0$

$\Rightarrow s = v^2/2g.$

Now consider the fragment flying off downward to the left.

$\updownarrow$

s	$v^2/2g$
u	$u\sin\theta$
v	x
a	g
t	t

$\leftrightarrow$

s	?
u	$u\cos\theta$
v	x
a	0
t	t

From the horizontal motion, $s_1 = ut\cos\theta \Rightarrow t = \frac{s_1}{u\cos\theta}$

From the vertical motion, $v^2/2g = ut\sin\theta + \frac{1}{2}gt^2$

Substituting the first equation into the second,

$$v^2/2g = s_1\tan\theta + \frac{1}{2}g\frac{s_1^2}{u^2\cos\theta}$$

$$\Rightarrow v^2 = 2gs_1\tan\theta + \frac{s_1^2 g^2}{u^2\cos^2\theta}$$

$$\Rightarrow s_1^2 g^2 + 2gs_1u^2\sin\theta\cos\theta - v^2u^2\cos^2\theta = 0$$

$$\Rightarrow s_1 = \frac{-2gu^2\sin\theta\cos\theta + \sqrt{4g^2u^4\sin^2\theta\cos^2\theta + 4g^2u^2v^2\cos^2\theta}}{2g^2}$$

$$= \frac{-2gu^2\sin\theta\cos\theta + 2gu\cos\theta\sqrt{u^2\sin^2\theta + v^2}}{2g^2}$$

$$= \frac{-u^2\sin\theta\cos\theta + u\cos\theta\sqrt{u^2\sin^2\theta + v^2}}{g}$$

For the fragment flying up and to the right,

s	$-v^2/2g$		s	$?$
u	$u\sin\theta$	$\leftrightarrow$	u	$u\cos\theta$
v	$\times$		v	$\times$
a	$-g$		a	0
t	t		t	t

Considering the horizontal motion, $s = u_2 t\cos\theta \Rightarrow t = \frac{s_2}{u\cos\theta}$

Considering the vertical motion, $-v^2/2g = ut\sin\theta - \frac{1}{2}gt^2$

$$\Rightarrow -v^2/2g = s_2\tan\theta - \frac{1}{2}g\frac{s_2^2}{u^2\cos^2\theta}$$

$$\Rightarrow s_2^2g^2 - 2s_2gu^2\sin\theta\cos\theta - v^2u^2\cos^2\theta$$

Note this is the same quadratic as before, but with the s_2 term with opposite sign.

So, $s_2 = \dfrac{u^2\sin\theta\cos\theta + u\cos\theta\sqrt{u^2\sin^2\theta+v^2}}{g}$

So $s_1+s_2 = \dfrac{2u\cos\theta\sqrt{u^2\sin^2\theta+v^2}}{g}$

Differentiating w.r.t θ and setting to zero,

$$\cancel{\frac{2u}{g}}\left(-\cancel{\sin\theta}\sqrt{u^2\sin^2\theta+v^2} + \frac{\cos\theta\cdot\frac{1}{2}\cdot 2u^2\cancel{\sin\theta}\cos\theta}{\sqrt{u^2\sin^2\theta+v^2}}\right) = 0$$

$\Rightarrow u^2\sin^2\theta+v^2 = u^2\cos^2\theta$

$\Rightarrow \cos^2\theta-\sin^2\theta = v^2/u^2$

$\Rightarrow \cos2\theta = v^2/u^2$ Note this has a solution only if $v\leq u$

$\Rightarrow 1-2\sin^2\theta = v^2/u^2$

$\Rightarrow \sin^2\theta = \dfrac{u^2-v^2}{2u^2}$

$\Rightarrow \cos\theta = \sqrt{1-\dfrac{u^2-v^2}{2u^2}}$

$= \sqrt{\dfrac{u^2+v^2}{2u^2}}$

So $s_1+s_2 = \dfrac{2u}{g}\cos\theta\sqrt{u^2\sin^2\theta+v^2}$

$= \dfrac{2u}{g}\sqrt{\dfrac{u^2+v^2}{2u^2}}\cdot\sqrt{\dfrac{u^2-v^2}{2}+v^2}$

$= \dfrac{2u}{g}\sqrt{\dfrac{u^2+v^2}{2u^2}}\cdot\sqrt{\dfrac{u^2+v^2}{2}}$

$= \dfrac{v^2+u^2}{g}$, as required.

If $u<v$, then s_1+s_2 has no extrema, so the maximum must occur at one of the endpoints. Clearly the maximum is not at $\theta=90°$, so must occur at $\theta=0$.

Then $s_1 + s_2 = \frac{2u}{g}\cos\theta\sqrt{u^2\sin^2\theta + v^2}$

$$= \frac{2uv}{g}, \text{ as required.}$$

Note that these solutions are the same when $u = v$.

STEP II 1994 Q12

$P(7 \text{ or } 11) = \frac{6}{36} + \frac{2}{36}$

$= \frac{8}{36}$

$= 2/9.$

P(wins on n^{th} throw)

$= \sum_{i=4,5,6,8,9,10}$ Throws i, then $(n-2)$ NOT i or 7, then i

$= 2\sum_{i=4,5,6}$ Throws i, then $(n-2)$ NOT i or 7, then i (because $P(4)=P(10)$, $P(5)=P(9)$, $P(6)=P(8)$)

$$= 2\left[\left(\frac{3}{36}\right)^2\left(1-\frac{3}{36}-\frac{6}{36}\right)^{n-2} + \left(\frac{4}{36}\right)^2\left(1-\frac{4}{36}-\frac{6}{36}\right)^{n-2} + \left(\frac{5}{36}\right)^2\left(1-\frac{5}{36}-\frac{6}{36}\right)^{2}\right]$$

$$= \frac{1}{72}\left(3/4\right)^{n-2} + \frac{2}{81}\left(\frac{13}{18}\right)^{n-2} + \frac{25}{648}\left(\frac{25}{36}\right)^{n-2} \qquad (*)$$

So P(wins on n^{th} throw | throws more than once)

$$= \frac{P(\text{wins on } n^{th} \text{ throw } (n>1))}{P(\text{throws more than once})}$$

$$= \frac{(*)}{P(4,5,6,8,9,10)}$$

$$= \frac{(*)}{2/3}$$

$$= \frac{1}{48}\left(\frac{3}{4}\right)^{n-2} + \frac{1}{27}\left(\frac{13}{18}\right)^{n-2} + \left(\frac{25}{432}\right)\left(\frac{25}{36}\right)^{n-2}$$, as required.

$P(\text{wins on } n^{th} \text{ throw} \mid > m \text{ throws})$

$$= \frac{P(\text{wins on } n^{th} \text{ throw } (n>m))}{P(> m \text{ throws})}$$

$$= \frac{\frac{1}{72}\left(3/4\right)^{n-2} + \frac{2}{81}\left(\frac{13}{18}\right)^{n-2} + \frac{25}{648}\left(\frac{25}{36}\right)^{n-2}}{2\left[\left(\frac{3}{36}\right)\left(3/4\right)^{m-1} + \left(4/36\right)\left(13/18\right)^{m-1} + \left(5/36\right)\left(25/36\right)^{m-1}\right.}$$

$$= \frac{\frac{1}{144}\left(\frac{3}{4}\right)^{n-2} + \frac{1}{81}\left(\frac{13}{18}\right)^{n-2} + \frac{25}{1296}\left(\frac{25}{36}\right)^{n-2}}{\frac{1}{12}\left(\frac{3}{4}\right)^{m-1} + \frac{1}{9}\left(\frac{13}{18}\right)^{m-1} + \frac{5}{36}\left(\frac{25}{36}\right)}$$

for $n > m$, and

$= 0$ for $n \leq m$.

STEP II 1994 Q13

The probability of the next card being new is $\frac{n-r}{n}$.

So $P($takes 1 card to get new card$) = \frac{n-r}{n}$

$P($ 2 cards $) = \left(\frac{r}{n}\right)\left(\frac{n-r}{n}\right)$

$P($ n cards $) = \left(\frac{r}{n}\right)^{n-1}\left(\frac{n-r}{n}\right)$

So the expectation is $\sum_{k=1}^{\infty} k\left(\frac{r}{n}\right)^{k-1}\left(\frac{n-r}{n}\right)$

Set $p = \frac{r}{n}$, $= \left(\frac{n-r}{n}\right)\sum_{k=1}^{\infty} kp^{k-1}$

$= \left(\frac{n-r}{n}\right)\sum_{k=1}^{\infty}\frac{d}{dp}(p^k)$

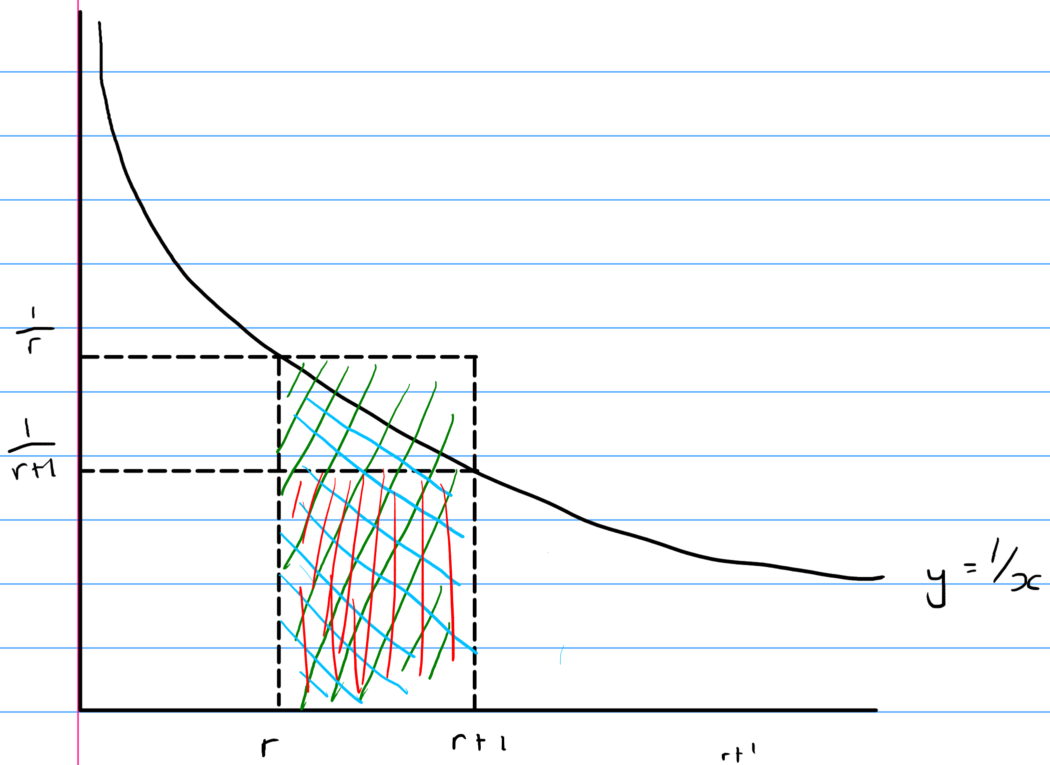
$= \left(\frac{n-r}{n}\right)\frac{d}{dp}\sum_{k=1}^{\infty}p^k$

$= \left(\frac{n-r}{n}\right)\frac{d}{dp}\frac{1}{1-p}$

$= \left(\frac{n-r}{n}\right)\cdot\frac{1}{(1-p)^2}$

$= \left(\frac{n-r}{n}\right)\cdot\left(\frac{n}{n-r}\right)^2$

$= \frac{n}{n-r}$, as required.



The green area is $1/r$, the blue area $\int_r^{r+1} \frac{1}{x}\,dx$, and the red area $\frac{1}{r+1}$.

So, $\frac{1}{r+1} \leq \int_r^{r+1} \frac{1}{x}\,dx \leq \frac{1}{r}$

Summing from 1 to $n-1$,

$$\sum_{r=1}^{n-1} \frac{1}{r+1} \leq \int_1^n \frac{1}{x}\,dx \leq \sum_{r=1}^{n-1} \frac{1}{r}$$

$$\Rightarrow \sum_{r=2}^{n} \frac{1}{r} \leq \ln n \leq \sum_{r=1}^{n-1} \frac{1}{r},$$ as required.

The expected time is $\sum_{r=0}^{n-1} \frac{n}{n-r}$

$$= n\sum_{r=0}^{n-1} \frac{1}{n-r}$$

$$= n\sum_{r=0}^{n} \frac{1}{n} = E$$

So using the inequality, $E - n \leq n\ln n \leq E - 1$

So for large n, $E \approx n\ln n$.

STEP II 1994 Q14

i) We pick a point at random on the board. Thus, the probability of a dart being distance r from the centre is proportional to $2\pi r$, the circumference of the circle through that point.

So, $f(r) = kr$.

We have $\int_0^a kr\,dr = 1$

$\Rightarrow \frac{1}{2}ka^2 = 1$

$\Rightarrow k = 2/a^2$

So $f(r) = \frac{2r}{a^2}$.

So $E(R) = \int_0^a \frac{2r^2}{a^2}\,dr = \left[\frac{2}{3a^2}r^3\right]_0^a$

$= \frac{2}{3}a$

$E(R^2) = \int_0^a \frac{2r^3}{a^2}\,dr = \left[\frac{1}{2a^2}r^4\right]_0^a$

$= \frac{1}{2}a^2$

So $\text{Var}\,R = ER^2 - (ER)^2$

$= \frac{1}{2}a^2 - \frac{4}{9}a^2$

$= a^2/18$.

ii) P(a dart being within $a/\sqrt{10}$)

$= \int_0^{a/\sqrt{10}} \frac{2}{a^2}r\,dr = \left[r^2/a^2\right]_0^{a/\sqrt{10}}$

$= \frac{1}{10}$

If m darts are thrown, $P(\text{within } 9/\sqrt{10} \text{ at least once})$

$$= 1 - P(\text{never within } 9/\sqrt{10})$$

$$= 1 - (9/10)^m$$

So expected loss is $(12+m) - 12\left(1-(9/10)^m\right)$

so $\frac{d(\text{loss})}{dm} = 1 + 12\left(\frac{9}{10}\right)^m \log\left(\frac{9}{10}\right) = 0$

$$\Rightarrow \left(\frac{9}{10}\right)^m = \frac{-1}{12\log(9/10)}$$

$$\Rightarrow m = \frac{1}{\log(9/10)} \log\left(\frac{-1}{12\log(9/10)}\right)$$

$$\approx 2.22$$

So the optimal number of darts must be either 2 or 3.

When $m=2$, expected loss is 11.72.
When $m=3$, expected loss is 11.74.

So choose $m=2$.