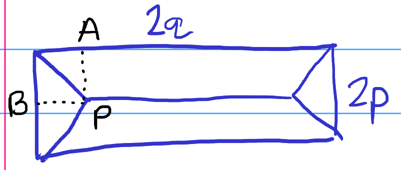
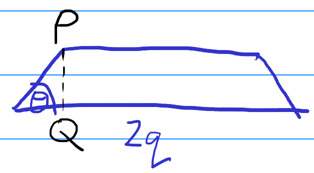
STEP I 1994 Question 1

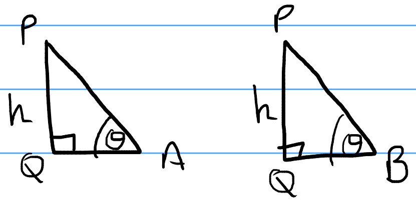
Plan view



Side view

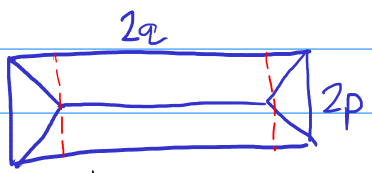


Point P is on the ridge line, Q is directly below it.
Consider triangles AQP and BQP.

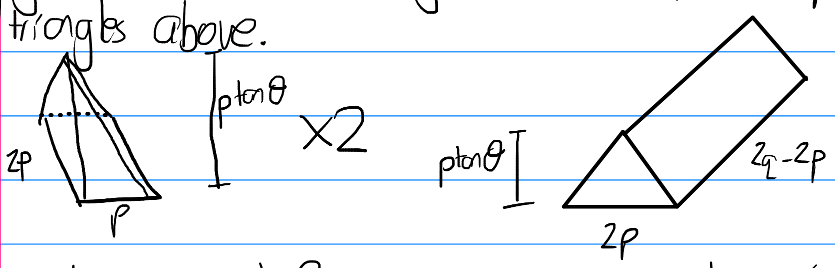


Both triangles are congruent, so $AP = BP = p$.
Thus the length of the ridge is $2q-2p$, which is independent of θ.

<u>Volume</u>



Divide the shape as shown, into a triangular prism and two identical pyramids. Note the height of the shape is $p\tan\theta$, clear from the right-angled triangles above.



$V = \frac{1}{3} \times 2p \times p \times p\tan\theta$
$= \frac{2}{3}p^3\tan\theta$

$V = \frac{1}{2} \times 2p \times p\tan\theta \times (2q-2p)$
$= 2p^2\tan\theta(q-p)$

$$V_{total} = \frac{4}{3}p^3\tan\theta + 2p^2\tan\theta(q-p)$$

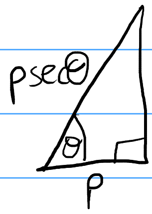
$= p^2\tan\theta\left(\frac{4}{3}p + 2q - 2p\right)$

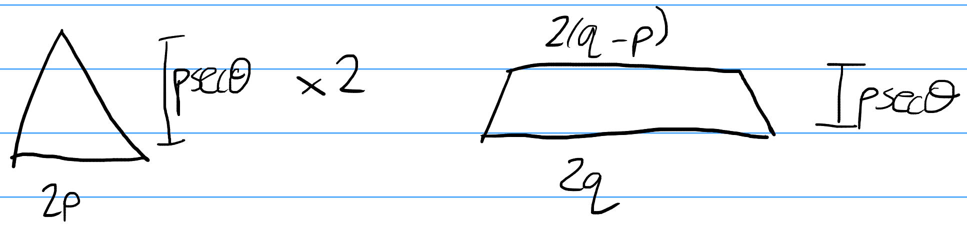
$= p^2\tan\theta\left(2q - \frac{2}{3}p\right)$

$= \frac{2}{3}p^2\tan\theta(3q - p)$

<u>Surface Area</u>

Split into the two trapezia and the two triangles. We need to find the height of these





$$SA_{total} = 2\times\frac{1}{2}\times 2p\times p\sec\theta + 2\times\frac{1}{2}(2q-2p+2q)p\sec\theta$$
$$= p\sec\theta(2p + 4q - 2p)$$
$$= 4pq\sec\theta$$

STEP I 1994 Question 2

i) $\frac{d}{dx}(x^a) = ax^{a-1}$

ii) $y = a^x$

$\ln y = x \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y \ln a = a^x \ln a$

iii) $y = x^x$

$\ln y = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$

$\frac{dy}{dx} = y(1 + \ln x)$

$= x^x(1+\ln x)$

iv) $y = x^{(x^x)}$

$\ln y = x^x \ln x$

$\frac{1}{y}\frac{dy}{dx} = x^x(1+\ln x)\ln x + \frac{x^x}{x}$

$\frac{dy}{dx} = yx^x\left(\ln x + (\ln x)^2 + \frac{1}{x}\right)$

$= x^{(x^x)} x^x \left(\ln x + (\ln x)^2 + \frac{1}{x}\right)$

v) $y = (x^x)^x$

$= x^{x \times x}$

$= x^{(x^2)}$

$\ln y = x^2 \ln x$

$\frac{1}{y}\frac{dy}{dx} = 2x\ln x + x$

$\frac{dy}{dx} = y\,x(2\ln x + 1)$

$= x^{x^2+1}(2\ln x + 1)$

STEP I 1994 Question 3

$(1-x)^n(1+x)^n \equiv (1-x^2)^n$

We consider the coefficient of x^n in the LHS. We can choose a term from the expansion of $(1-x)^n$, say the x^k term. This is $(-1)^n\binom{n}{k}$. To make x^n, we need the x^{n-k} term from $(1+x)^n$, which has coefficient $\binom{n}{n-k}=\binom{n}{k}$. So, the coefficient when combining these is

$$(-1)^n\binom{n}{k}\binom{n}{k} = (-1)^n\binom{n}{k}^2.$$

Summing over all the choices for k, we obtain

$$\binom{n}{0}^2-\binom{n}{1}^2+\binom{n}{2}^2+\ldots+(-1)^n\binom{n}{n}^2 \qquad (*)$$

Now considering the RHS, $(1-x^2)^n$.

When n is even, the x^n term comes from $(x^2)^{n/2}$ which has coefficent $(-1)^{n/2}\binom{n}{n/2}$.
So, when n is even, $(*)=(-1)^{n/2}\binom{n}{n/2}$.

The expansion only generates even powers of x, so when n is odd the coefficient of x^n is 0, so $(*)=0$.

STEP I 1994 Q4

(i) $\dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{1-\cos(2\times\frac{\alpha}{2})}{\sin(2\times\frac{\alpha}{2})}$

$= \dfrac{1-(1-2\sin^2\alpha/2)}{2\sin\alpha/2\cos\alpha/2}$

$= \dfrac{2\sin^2\alpha/2}{2\sin\alpha/2\cos\alpha/2}$

$= \tan\alpha/2 \qquad \square$

(ii) $\displaystyle\int \frac{1}{1-2kx+x^2}\,dx$

$\displaystyle= \int \frac{1}{(x-k)^2+1-k^2}\,dx$

$\displaystyle= \frac{1}{\sqrt{1-k^2}}\arctan\left(\frac{x-k}{\sqrt{1-k^2}}\right)+C \qquad (*)$

$\displaystyle\int_0^1 \frac{\sin\alpha}{1-2x\cos\alpha+x^2}$

We use $(*)$ with $k=\cos\alpha$, and note that $\sin\alpha$ is a constant, to obtain

$\displaystyle\left[\frac{\sin\alpha}{\sqrt{1-\cos^2\alpha}}\arctan\left(\frac{x-\cos\alpha}{\sqrt{1-\cos^2\alpha}}\right)\right]_0^1$

Now, $\sqrt{1-\cos^2\alpha}=\sin\alpha$, so this equals

$\displaystyle\left[\arctan\left(\frac{x-\cos\alpha}{\sin\alpha}\right)\right]_0^1$

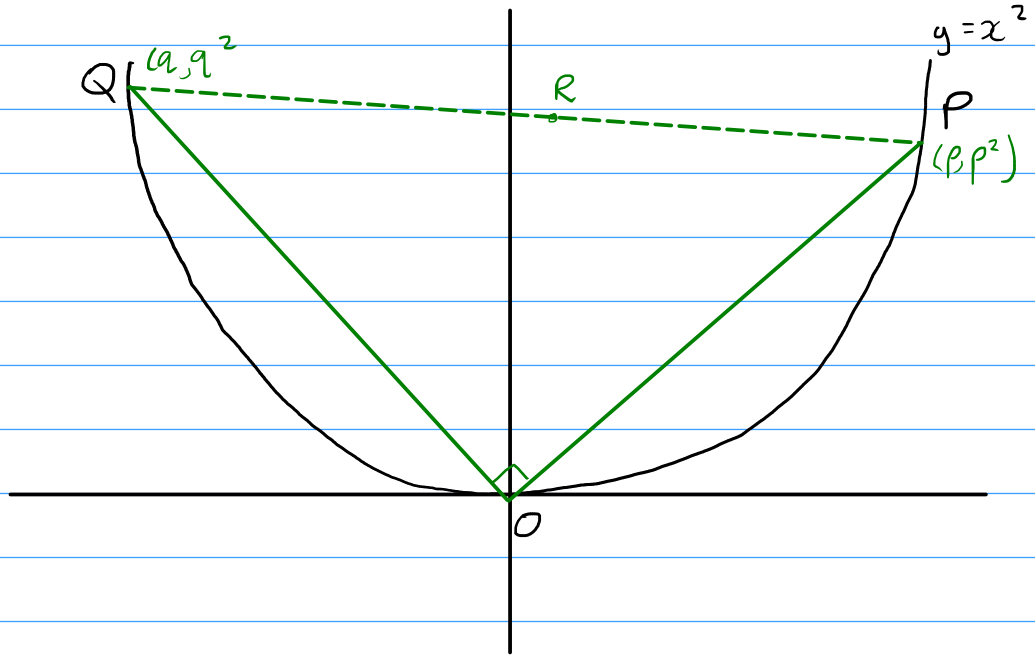
$\displaystyle= \arctan\left(\frac{1-\cos\alpha}{\sin\alpha}\right) - \arctan\left(\frac{-\cos\alpha}{\sin\alpha}\right)$

By part (i), $\frac{1-\cos\alpha}{\sin\alpha} = \tan\alpha/2$, and $\arctan\left(\frac{-1}{\tan\alpha}\right) = \alpha - \pi/2$ (consider the graph), so

$\alpha/2 - (\alpha - \pi/2)$

$= \frac{1}{2}(\pi - \alpha)$ $\square$

STEP I 1994 Q5



i) The gradient of OP is $\frac{p^2}{p}=p$, and OP and OQ are perpendicular, so the gradient of OQ is $\frac{-1}{p}$. Thus $\frac{q^2}{q}=\frac{-1}{p} \Rightarrow q=-1/p$.

So the coordinates of Q is $\left(\frac{-1}{p}, \frac{1}{p^2}\right)$.

Thus the coordinates of R, the midpoint of PQ, is

$$\left(\tfrac{1}{2}(p-\tfrac{1}{p}),\ \tfrac{1}{2}(p^2+1/p^2)\right)$$

$$=\left(\frac{p^2-1}{2p}, \frac{p^4+1}{2p^2}\right)=(x,y)$$

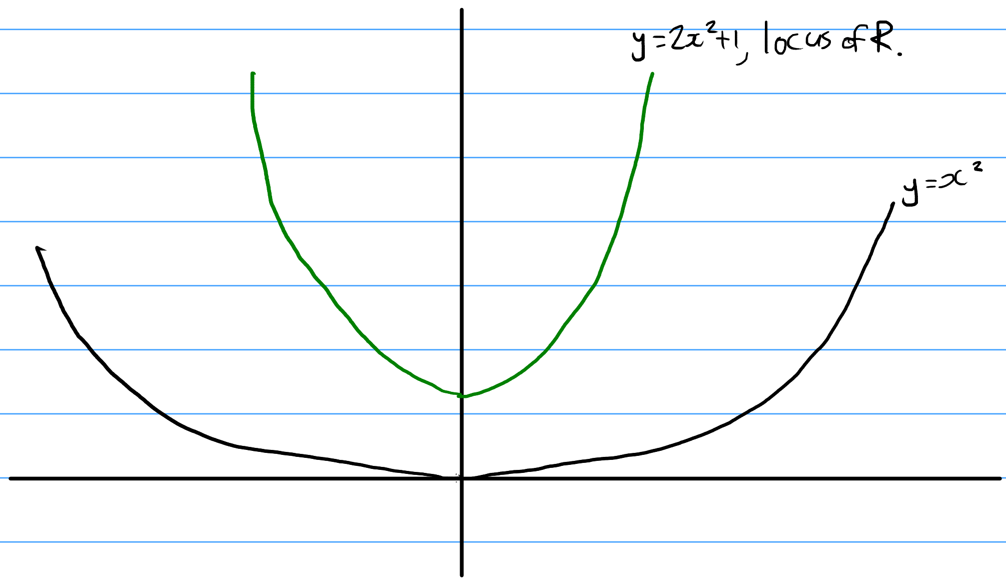
So $x^2=\frac{(p^2-1)^2}{4p^2}$

Thus $2x^2-y=\frac{(p^2-1)^2}{2p^2}-\frac{p^4+1}{2p^2}$

$$=\frac{p^4-2p^2+1-p^4-1}{2p^2}$$

$$=\frac{-2p^2}{2p^2}=-1$$

Thus we have $2x^2 - y = -1$, so $y = 2x^2 + 1$.



ii) $\frac{dy}{dx} = 2x$.

So the tangent to the curve at P is

$$y - p^2 = 2p(x - p)$$

$$\Rightarrow y = 2px - p^2$$

The tangent to the curve at Q is

$$y - \frac{1}{p^2} = \frac{-2}{p}\left(x + \frac{1}{p}\right)$$

$$\Rightarrow y = \frac{-2}{p}x - \frac{1}{p^2}$$

Setting these equal to each other,

$$2px - p^2 = \frac{-2}{p}x - \frac{1}{p^2}$$

$$\Rightarrow x(2p + 2/p) = p^2 - \frac{1}{p^2}$$

$$\Rightarrow 2x\left(\frac{p^2+1}{p}\right) = \frac{p^4 - 1}{p^2}$$

$$\Rightarrow x = \frac{p^4 - 1}{2p(p^2+1)}$$

$$x = \frac{p^4-1}{2p(p^2+1)}$$

$$= \frac{(p^2+1)(p^2-1)}{2p(p^2+1)}$$

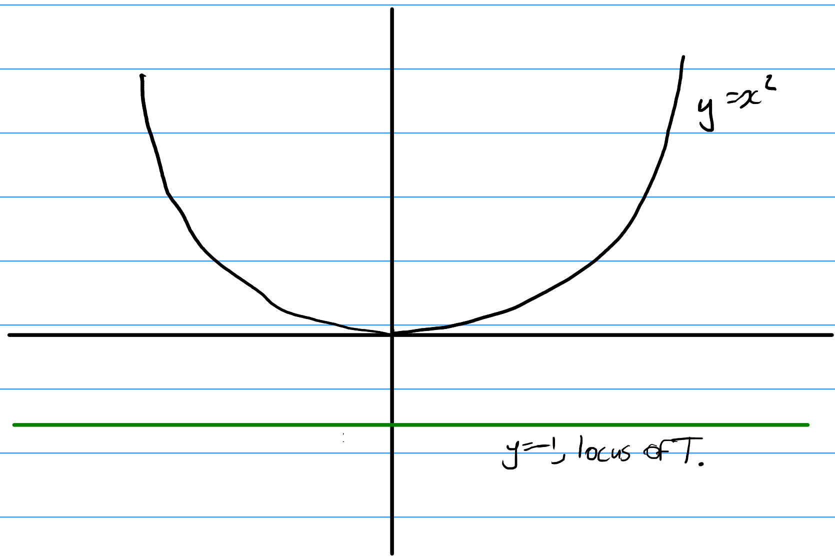
$$= \frac{p^2-1}{2p}$$

Now $y = 2px - p^2$ so $y = 2p\frac{p^2-1}{2p} - p^2$

$$= p^2 - 1 - p^2$$

$$= -1$$

So the locus of T is $\left(\frac{p^2-1}{2p}, -1\right)$ as p varies. This is just the line $y=-$



STEP I 1994 Q6

$$f(z) = \frac{iz-1}{iz+1}$$

i) $$f(x) = \frac{ix-1}{ix+1} \times \frac{1-ix}{1-ix}$$

$$= \frac{-(1-ix)^2}{1+x^2}$$

$$= \frac{x^2-1+2ix}{1+x^2}$$

So $\mathrm{Re} f(x) = \frac{x^2-1}{1+x^2}$ $\mathrm{Im} f(x) = \frac{2x}{1+x^2}$

ii) $f(x)f(x)^*$

$$= |\mathrm{Re} f(x)|^2 + |\mathrm{Im} f(x)|^2$$

$$= \frac{(x^2-1)^2+(2x)^2}{(1+x^2)^2}$$

$$= \frac{x^4-2x^2+1+4x^2}{(1+x^2)^2}$$

$$= \frac{x^4+2x^2+1}{x^4+2x^2+1}$$

$$= 1$$

iii) $$ff(x) = \frac{i\frac{ix-1}{ix+1}-1}{i\frac{ix-1}{ix+1}+1}$$

$$= \frac{i(ix-1)-(ix+1)}{i(ix-1)+(ix+1)}$$

$$= \frac{-x-i-ix-1}{-x-i+ix+1}$$

$$ff(x) = \frac{-x-i-ix-1}{-x-i+ix+1}$$

$$= \frac{(-x-1)+i(-x-1)}{(-x+1)+i(x-1)} \times \frac{(-x+1)-i(x-1)}{(-x+1)-i(x-1)}$$

$$= \frac{(-x-1)(-x+1)-i(-x-1)(x-1)+i(-x-1)(-x+1)-i^2(-x-1)(x-1)}{(-x+1)^2+(x-1)^2}$$

The first and last terms cancel, so we get

$$ff(x) = \frac{2i(x+1)(x-1)}{2(x-1)^2}$$

$$= i\,\frac{x+1}{x-1}$$

So $\mathrm{Re}\,ff(x) = 0$, $\mathrm{Im}\,ff(x) = \frac{x+1}{x-1}$

iv) $fff(x) = f(ff(x))$

$$= \frac{-\left(\frac{x+1}{x-1}\right)-1}{-\left(\frac{x+1}{x-1}\right)+1}$$

$$= \frac{-x-1-x+1}{-x-1+x-1}$$

$$= \frac{-2x}{-2}$$

$$= x$$

So $fff(x) = x$.

STEP I 1994 Q7

Conjecture: $\displaystyle\sum_{i=k^2+1}^{(k+1)^2} i = k^3+(k+1)^3$

Proof: $\displaystyle\sum_{i=k^2+1}^{(k+1)^2} i = \sum_{i=1}^{(k+1)^2} i - \sum_{i=1}^{k^2} i$

$$= \tfrac{1}{2}(k+1)^2\left[(k+1)^2+1\right] - \tfrac{1}{2}k^2(k^2+1)$$

$$= \tfrac{1}{2}\left[(k^2+2k+1)(k^2+2k+2) - k^4 - k^2\right]$$

$$= \tfrac{1}{2}\left[k^4+4k^3+7k^2+6k+2-k^4-k^2\right]$$

$$= \tfrac{1}{2}\left[4k^3+6k^2+6k+2\right]$$

$$= 2k^3+3k^2+3k+1$$

$$= k^3+(k^3+3k^2+3k+1)$$

$$= k^3+(k+1)^3 \text{, as required.}$$

$$1^3+2^3+3^3+4^3+\underline{\qquad\qquad}+N^3$$

$$= \tfrac{1}{2}\left[(0^3+1^3)+(1^3+2^3)+(2^3+3^3)+\underline{\qquad\quad}+((N-1)^3+N^3)+N^3\right]$$

$$= \tfrac{1}{2}\left[1+2+3+4+\underline{\qquad\qquad}+N^2+N^3\right]$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}N^2(N^2+1)+N^3\right]$$

$$= \tfrac{1}{4}N^2(N^2+1+2N)$$

$$= \tfrac{1}{4}N^2(N+1)^2 \text{, as required.}$$

STEP I 1994 Q8

$I = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$ $\theta = \frac{\pi}{4} - \phi$

so $\phi = \pi/4 - \theta$ and $d\theta = -d\phi$

$= \int_{\pi/4}^{0} -\ln(1+\tan(\pi/4 - \phi))\, d\phi$

$= \int_0^{\pi/4} \ln(1+\tan(\pi/4-\phi))\, d\phi$ $(*)$

Now $1+\tan(\pi/4-\phi)$

$= 1 + \frac{\tan\pi/4 - \tan\phi}{1+\tan\pi/4 \tan\phi}$

$= 1 + \frac{1-\tan\phi}{1+\tan\phi}$

$= \frac{1+\tan\phi + 1 - \tan\phi}{1+\tan\phi}$

$= \frac{2}{1+\tan\phi}$

So $(*)$ becomes $\int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\phi}\right) d\phi$

$= \int_0^{\pi/4} \ln 2 - \ln(1+\tan\phi)\, d\phi$

$= \int_0^{\pi/4} \ln 2\, d\phi - I$

so $2I = \int_0^{\pi/4} \ln 2\, d\phi$

$= \frac{\pi}{4}\ln 2$

$\Rightarrow I = \frac{\pi}{8}\ln 2$ $\square$

$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx$ Set $x = \tan\theta$, so $\frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\, d\theta$

So the integral becomes $\int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta} \sec^2\theta\, d\theta$, but $1+\tan^2\theta = \sec^2\theta$ so the integral becomes the same as the one above, so has value $\frac{\pi}{8}\ln 2$.

$$\int_0^{\pi/2} \ln\left(\frac{1+\sin x}{1+\cos x}\right) dx$$

$$= \int_0^{\pi/2} \ln(1+\sin x) - \ln(1+\cos x)\, dx$$

$$= \int_0^{\pi/2} \ln(1+\sin x)\, dx - \int_0^{\pi/2} \ln(1+\sin(\pi/2 - x))\, dx$$

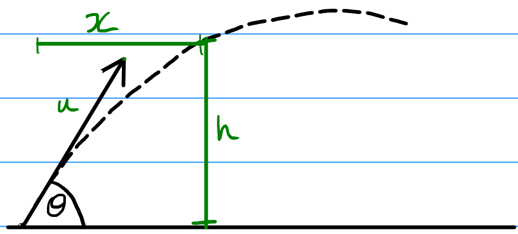
set $u = \pi/2 - x$, $du = -dx$

$$= \int_0^{\pi/2} \ln(1+\sin x)\, dx - \int_{\pi/2}^{0} -\ln(1+\sin u)\, du$$

$$= \int_0^{\pi/2} \ln(1+\sin x)\, dx - \int_0^{\pi/2} \ln(1+\sin u)\, du$$

$$= 0$$

1994 STEP I Q9



↕	
s	h
u	$u\sin\theta$
v	×
a	$-g$
t	t

⟷	
s	x
u	$u\cos\theta$
v	$u\cos\theta$
a	0
t	t

The horizontal motion has constant speed, so $x = ut\cos\theta \Rightarrow \cos\theta = \frac{x}{ut}$ $(*)$

For the vertical motion, $s = ut + \frac{1}{2}at^2$, so $h = ut\sin\theta - \frac{1}{2}gt^2 \Rightarrow \sin\theta = \frac{h + \frac{1}{2}gt^2}{ut}$ $(\dagger)$

Now, using $\cos^2\theta + \sin^2\theta = 1$, and $(*)$ and $(\dagger)$, we obtain

$$\left(\frac{x}{ut}\right)^2 + \left(\frac{h + \frac{1}{2}gt^2}{ut}\right)^2 = 1$$

$$\Rightarrow x^2 + h^2 + hgt^2 + \frac{1}{4}g^2t^4 = u^2t^2$$

$$\Rightarrow \frac{1}{4}g^2t^4 - (u^2 - gh)t^2 + h^2 + x^2 = 0, \text{ as required.}$$

Now, we can consider this as a quadratic in t^2. The equation come from a point on the trajectory, so there must be real solutions for t, thus the discriminant is positive. Hence,

$$(u^2 - gh)^2 - g^2(h^2 + x^2) \geq 0$$

$$\Rightarrow u^4 - 2u^2gh + \cancel{g^2h^2} - \cancel{g^2h^2} - g^2x^2 \geq 0$$

$$\Rightarrow u^2(u^2 - 2gh) \geq g^2x^2$$

$$\Rightarrow x \leq \frac{u}{g}\sqrt{u^2 - 2gh} \text{, as required.}$$

Now we want to show there exists θ such that this range is attained. So we reintroduce θ.

From the top, $x = ut\cos\theta \Rightarrow t = \frac{x}{u\cos\theta} = \frac{\frac{u}{g}\sqrt{u^2 - 2gh}}{u\cos\theta} = \frac{\sqrt{u^2 - 2gh}}{g\cos\theta}$.

We also have $h = ut\sin\theta - \frac{1}{2}gt^2$.

Substituting the first equation into the second,

$$u\sin\theta \cdot \frac{\sqrt{u^2-2gh}}{g\cos\theta} - \frac{1}{2}g \cdot \frac{u^2-2gh}{g^2\cos^2\theta} = h$$

$$\Rightarrow \frac{u}{g}\sqrt{u^2-2gh}\tan\theta - \frac{u^2-2gh}{2g\cos^2\theta} = h$$

$$\Rightarrow \frac{u^2-2gh}{2g}\sec^2\theta - \frac{u}{g}\sqrt{u^2-2gh}\tan\theta + h = 0$$

But $\sec^2\theta = 1 + \tan^2\theta$, so

$$\frac{u^2}{2g} - h + \frac{u^2-2gh}{2g}\tan^2\theta - \frac{u}{g}\sqrt{u^2-2gh}\tan\theta + h = 0$$

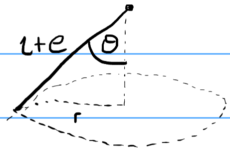
$$\Rightarrow \frac{u^2-2gh}{2g}\tan^2\theta - \frac{u}{g}\sqrt{u^2-2gh}\tan\theta + \frac{u^2}{2g} = 0$$

$$\Rightarrow (u^2-2gh)\tan^2\theta - 2u\sqrt{u^2-2gh}\tan\theta + u^2 = 0$$

For a solution for $\tan\theta$ to exist, we need the discriminant to be non-negative (note any real value for $\tan\theta$ gives a value for θ).

$\Delta = 4u^2(u^2-2gh) - 4(u^2-2gh)\cdot u^2 = 0$, so a solution exists for $\tan\theta$ and hence for θ.

STEP I 1994 Q10



l is natural length, e is extension

$\updownarrow \quad mg = \frac{\lambda e}{l}\cos\theta \qquad (1)$

$\leftrightarrow \quad \frac{\lambda e}{l}\sin\theta = mr\omega^2 \qquad (2)$

But $\sin\theta = \frac{r}{l+e}$, so (2) becomes

$$\frac{\lambda e}{l}\cdot\frac{\cancel{r}}{l+e} = m\cancel{r}\omega^2$$

$$\Rightarrow \frac{\lambda e}{l(l+e)} = m\omega^2$$

$$\Rightarrow \lambda e = m\omega^2 l^2 + m\omega^2 le$$

$$\Rightarrow e = \frac{m\omega^2 l^2}{\lambda - m\omega^2 l}$$

Substituting this into (1),

$$\cancel{m}g = \frac{\lambda}{\cancel{l}}\cdot\frac{\cancel{m}\omega^2 l^{\cancel{2}}}{\lambda - m\omega^2 l}\cos\theta$$

$$\text{So } \cos\theta = \frac{g(\lambda - m\omega^2 l)}{\lambda\omega^2 l}$$

Now $0<\theta<\pi/2 \Rightarrow 0<\cos\theta<1$.

$\cos\theta > 0 \Rightarrow \lambda > m\omega^2 l$

$\Rightarrow \omega^2 < \frac{\lambda}{ml}$

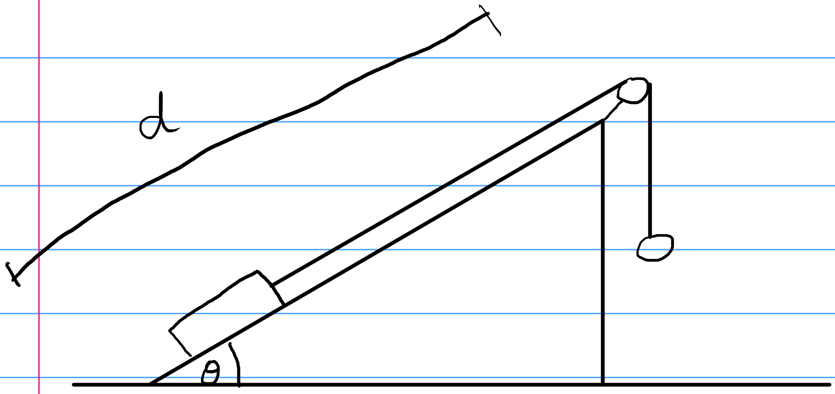
$\cos\theta < 1 \Rightarrow g(\lambda - m\omega^2 l) < \lambda l\omega^2$

$\Rightarrow \omega^2(\lambda l + gml) > g\lambda$

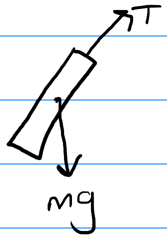
$\Rightarrow \omega^2 > \frac{g\lambda}{l(\lambda+mg)}$.

So, $\frac{g\lambda}{l(\lambda+mg)} < \omega^2 < \frac{\lambda}{ml}$, as required.

STEP I 1994 Q11



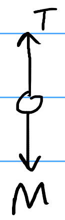
i) For the wagon,



so $T - mg\sin\theta = ma$

$\Rightarrow T = ma + mg\sin\theta$

For the ball



$Mg - T = Ma$

$\Rightarrow T = Mg - Ma$

Combining these, $ma + mg\sin\theta = M - Ma$, so

$$a = \frac{Mg - mg\sin\theta}{m + M}$$

For the ball will fall to the ground we must have $a > 0$, so

$$\frac{Mg - mg\sin\theta}{m + M} > 0$$

or $M - m\sin\theta > 0$ or $\sin\theta < \frac{M}{m}$

using suvat,

s	$d\sin\theta$
u	0
v	$?$
a	$\frac{Mg - mg\sin\theta}{M+m}$
t	$\times$

$$v^2 = 2as = \frac{2dg\sin\theta\,(M - m\sin\theta)}{M+m}$$, as required.

(ii) The wagon has already travelled $d\sin\theta$. After the ball hits the ground, T becomes 0, so now the equation of motion for the wagon is

$$-m\sin\theta = ma$$
$$\Rightarrow a = -\sin\theta.$$

s	$?$
u	$\sqrt{v^2}$ (above)
v	0
a	$-\sin\theta$
t	$\times$

$$v^2 = u^2 + 2as$$

$$\Rightarrow \frac{\cancel{2}g(M - m\sin\theta)d\cancel{\sin}\theta}{M+m} - \cancel{2}s\cancel{\sin}\theta = 0$$

$$\Rightarrow s = \frac{gd(M - m\sin\theta)}{M+m}$$

So

$$S_{total} = \cancel{d}\sin\theta + \frac{g\cancel{d}(M - m\sin\theta)}{M+m} < \cancel{d}$$

$$\Rightarrow \frac{g(M - m\sin\theta)}{M+m} < 1 - \sin\theta$$

STEP I 1994 Q12

i) $\frac{2}{28} = \frac{1}{14}$

ii) P(picked Newnham) = P(picked first) + P(picked second) + P(picked third)

$$= \frac{1}{28} + \frac{\cancel{27}}{28} \times \frac{1}{\cancel{27}} + \frac{\cancel{27}}{28} \times \frac{\cancel{26}}{\cancel{27}} \times \frac{1}{\cancel{26}}$$

$$= 3/28$$

iii) P(Newnham or New Hall or both)

$= 1 - P(\text{neither})$

$= 1 - \frac{26}{28} \times \frac{25}{27} \times \frac{24}{26}$

$= 1 - \frac{50}{63}$

$= 13/63$

iv) $1 - P(\text{both are not New Hall})$

$= 1 - \frac{26}{27} \times \frac{25}{26}$

$= 1 - \frac{25}{27}$

$= 2/27$

v) $P(\text{New Hall} \mid \text{Newnham}) = \dfrac{P(\text{New Hall} \cap \text{Newnham})}{P(\text{Newnham})}$

$P(\text{New Hall} \cap \text{Newnham}) = P(WWB) + P(WBW) + P(BWW)$

$$= \frac{2}{28} \times \frac{1}{27} \times 1 + \frac{2}{28} \times \frac{26}{27} \times \frac{1}{26} + \frac{26}{28} \times \frac{2}{27} \times \frac{1}{26}$$

$$= \frac{6}{27 \times 28}$$

So $P(\text{New Hall} \mid \text{Newnham}) = \dfrac{\frac{6}{27 \times 28}}{\frac{3}{28}} = \frac{2}{27}$

vi) This is the same as part (iv), as none of the probabilities depend on which women's college is picked first, so the answer is $2/27$.

vii) $$P(2 \text{ single sex} \mid \geq 1 \text{ single sex}) = \frac{P(2 \text{ single sex} \cap \geq 1 \text{ single sex})}{P(\geq 1 \text{ single sex})}$$

$$= \frac{P(2 \text{ single sex})}{P(\geq 1 \text{ single sex})}$$

$$= \frac{\frac{6}{27 \times 28}}{13/63} = \frac{2 \times 3}{3^3 \times 2^2 \times 7} \times \frac{3^2 \times 7}{13}$$

$$= \frac{1}{26}$$

STEP I 1994 Q13

i) $X_i = \begin{cases} 1 & i^{th} \text{ token is red} \\ 0 & i^{th} \text{ token is not red} \end{cases}$, so $X = X_1 + X_2 + \dots + X_n$ is the number of tokens which are red.

ii) $EX_i = P(X_i = 1)$

$= P(i^{th} \text{ token is red}).$

Now, before drawing any tokens, the probability of each one being red must be the same as there is no reason why any particular draw would be more ~~or~~ less likely.

So, $EX_i = EX_1$

$= P(X_1 = 1)$

$= \frac{m}{M}$

iii) $EX = E\sum_{i=1}^{n} X_i$

$= \sum_{i=1}^{n} EX_i$ by linearity of expectation.

$= \sum_{i=1}^{n} \frac{m}{M}$

$= \frac{mn}{M}$

iv) $P(X=k) = \frac{\text{number of useful outcomes}}{\text{total number of outcomes}}$

The total number of outcomes is $\binom{M}{n}$

The number of useful outcomes is $\binom{m}{k}\binom{M-m}{n-k}$

choosing k reds from m

Choosing $(n-k)$ non-reds from $(M-m)$

So, $P(X=k) = \dfrac{\binom{m}{k}\binom{M-m}{n-k}}{\binom{M}{n}}$

v) $EX = \sum_{k=1}^{n} k P(X=k)$

so $\sum_{k=1}^{n} k \dfrac{\binom{m}{k}\binom{M-m}{n-k}}{\binom{M}{n}} = \dfrac{mn}{M}$

$\Rightarrow \sum_{k=1}^{n} k\binom{m}{k}\binom{M-m}{n-k} = \dfrac{mn}{M}\binom{M}{n}$

$= \dfrac{M!}{n!(M-n)!}\,\dfrac{mn}{M}$

$= \dfrac{m(M-1)!}{(n-1)!(M-n)!}$

$= m\binom{M-1}{n-1}$, as required.

STEP I 1994 Q14

i) $f(t) = \lambda e^{-\lambda t}$

$ET_i = \int_0^\infty \lambda t e^{-\lambda t}\, dt$ $\quad u\ t \quad v'\ \lambda e^{-\lambda t}$

$\quad u'\ 1 \quad v\ -e^{-\lambda t}$

$= \left[-te^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}\, dt$

$= 0 + \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty$

$= 1/\lambda$

$ET_i^2 = \int_0^\infty \lambda t^2 e^{-\lambda t}\, dt$ $\quad u\ t^2 \quad v'\ \lambda e^{-\lambda t}$

$\quad u'\ 2t \quad v\ -e^{-\lambda t}$

$= \left[-t^2 e^{-\lambda t}\right]_0^\infty + \frac{2}{\lambda}\int_0^\infty \lambda t e^{-\lambda t}\, dt$

$= 0 + 2/\lambda^2$

so $\text{Var}\,T_i = ET_i^2 - (ET_i)^2$

$= 2/\lambda^2 - 1/\lambda^2$

$= 1/\lambda^2$

ii) $P(U \le u) = P(T_1 \le u \cap T_2 \le u \cap \dots \cap T_n \le u)$

$= P(T_1 \le u)P(T_2 \le u)\dots P(T_n \le u)$ by independence.

$= P(T_1 \le u)^n$ as they have the same distribution.

$P(T_i \le u) = \int^u \lambda e^{-\lambda t}\, dt$

$= -e^{-\lambda u} + c$

$F(\infty) = 1 \Rightarrow c = 1$, so $P(T_i \le u) = 1 - e^{-\lambda u}$

So $P(U \le u) = (1 - e^{-\lambda u})^n$

Density function for U is $\frac{d}{du}(1-e^{-\lambda u})^n$

$$= n(1-e^{-\lambda u})^{n-1} \times \lambda e^{-\lambda u}$$

$$= n\lambda e^{-\lambda u}(1-e^{-\lambda u})^{n-1}$$

iii) $P(T>t) = P(T_1 \geq t \cap T_2 \geq t \cap T_3 \geq t \cap \dots \cap T_n \geq t)$

$$= P(T_1 \geq t)^n \qquad \text{as before}$$

$$= (e^{-\lambda t})^n$$

$$= e^{-\lambda n t}$$

So $P(T<t) = 1 - e^{-n\lambda t}$

Density function for T is $\frac{d}{dt}(1-e^{-n\lambda t})$

$$= n\lambda e^{-n\lambda t}$$

iv) T has the same distribution as the T_i, but with parameter $n\lambda$, rather than λ.

So, $ET = \frac{1}{n\lambda}$, $\mathrm{Var}\,T = \frac{1}{n^2\lambda^2}$